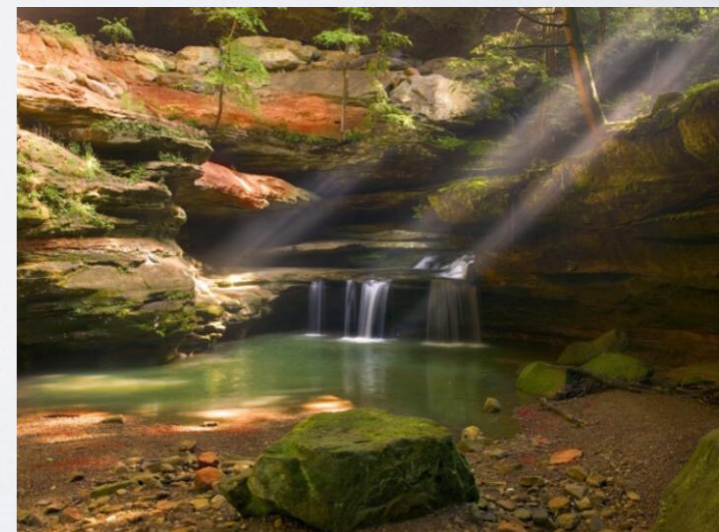


# THE WELLSPRING OF ALL PHASES ON THE KAGOME LATTICE



University of Illinois at Urbana Champaign

with Hitesh Changlani, Dmitrii Kochkov, Krishna Kumar, Eduardo Fradkin



## **Other talks I didn't give that I'm happy to chat about.**

*An inverse approach to strongly correlated systems*

*Many-Body Localization and Holography*

*Beyond many-body localization: eigenstates with logarithmic entanglement.*

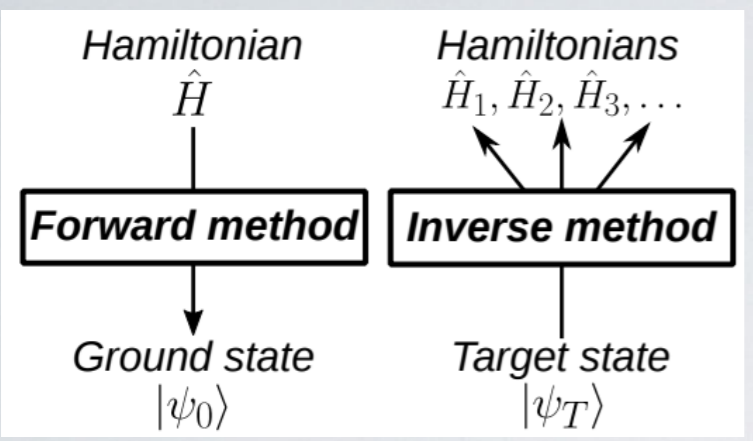
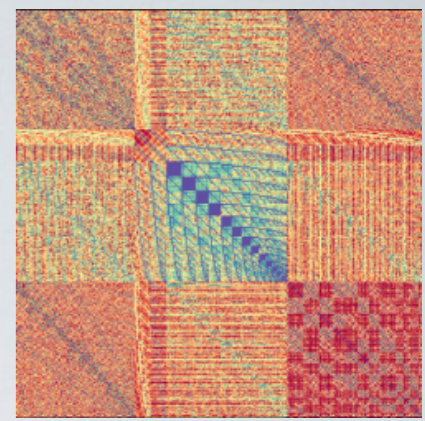
*Finite Temperature Variational Monte Carlo*

*Wave-functions from Deep Neural Nets (without the hype)*

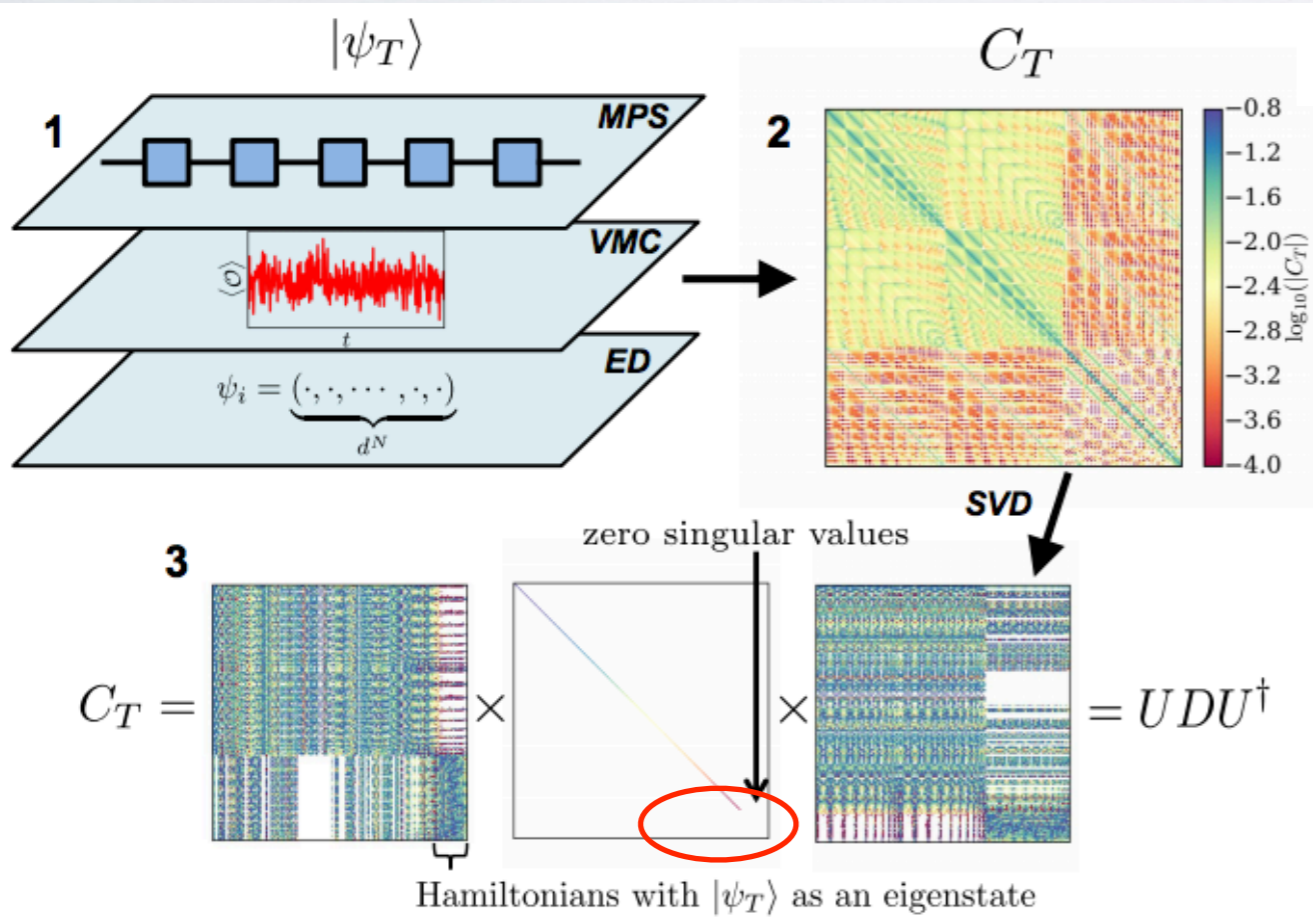
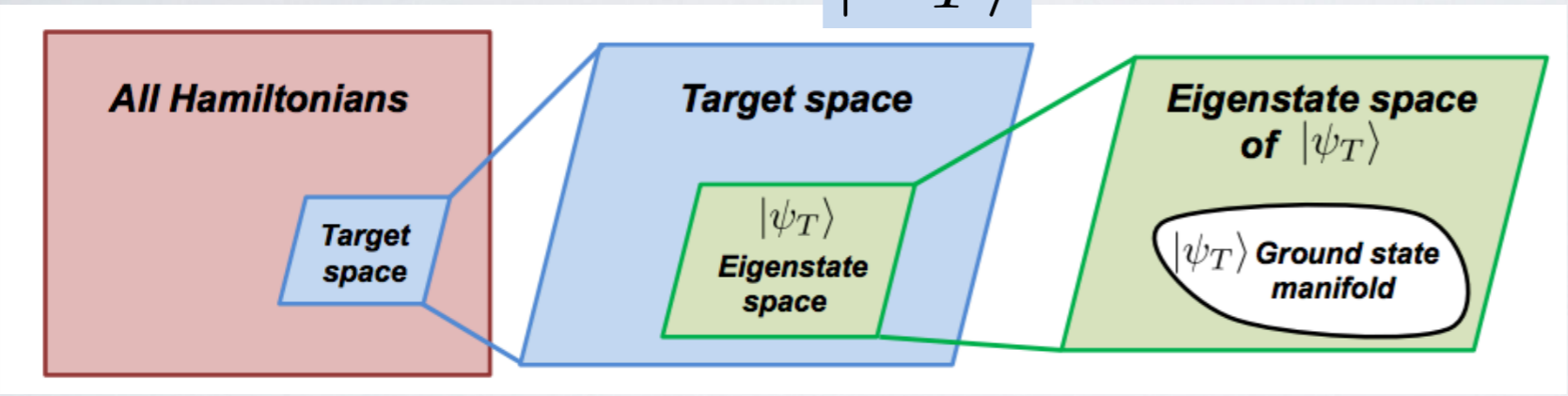


But first an aside...

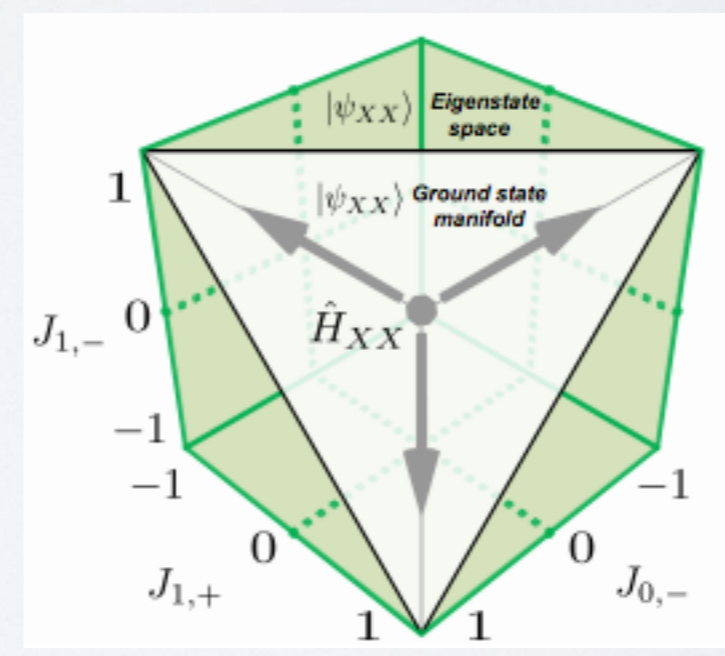
Inverse design in strongly correlated systems



$|\Psi_T\rangle$



$(C_T)_{ab} = \langle \hat{h}_a \hat{h}_b \rangle_T - \langle \hat{h}_a \rangle_T \langle \hat{h}_b \rangle_T$



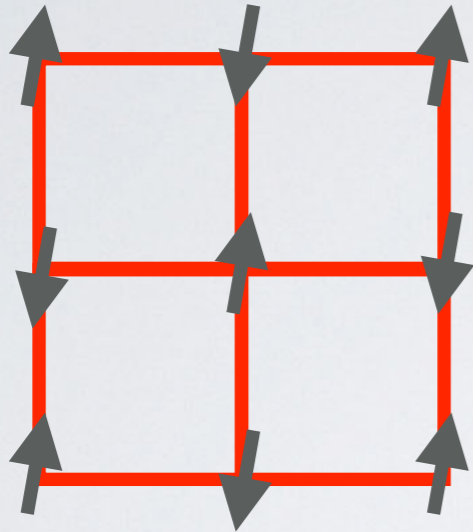
Eli Chertkov



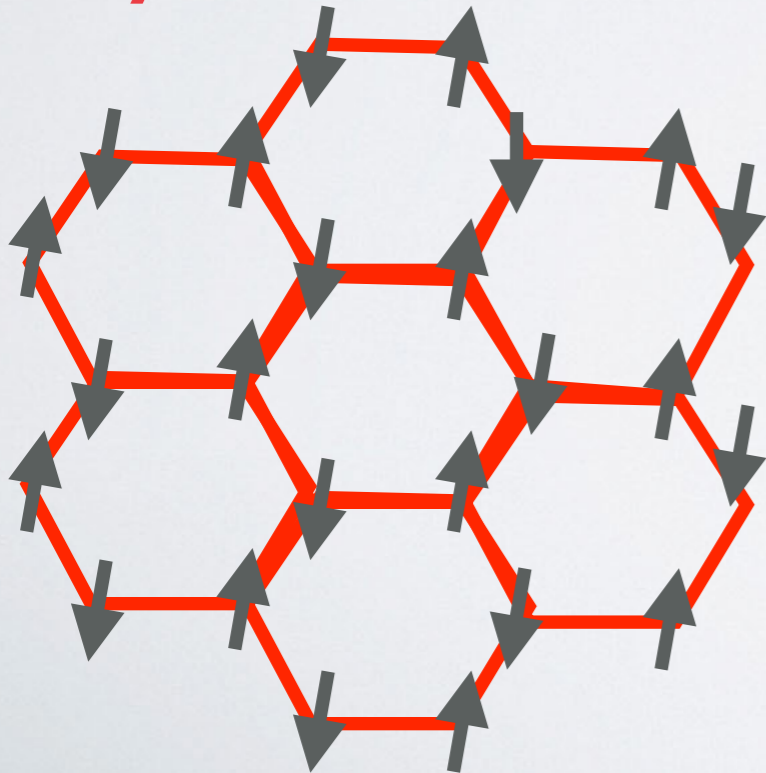
# Quantum Materials.....

Insulator - Electrons don't move  
Interaction between electron spins

## Square Lattice



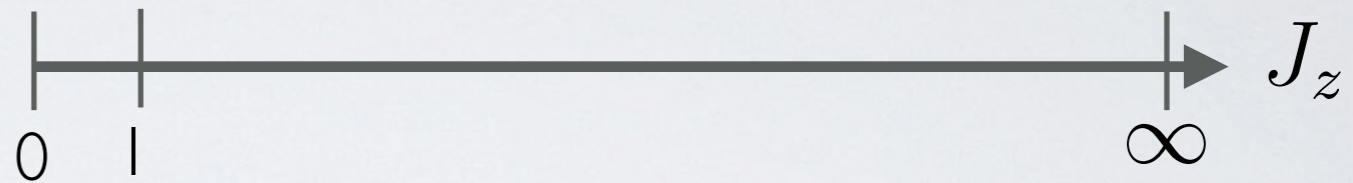
## Honeycomb Lattice



$$H_{xy} = \sum_{ij} S_i^x S_j^x + S_i^y S_j^y$$

$$H_{xxz} = H_{xy} + J_z \sum_{ij} S_i^z S_j^z$$

$$H_{\text{ising}} = \sum_{ij} S_i^z S_j^z \quad (\text{ising limit } J_z \rightarrow \infty)$$



*Anti-ferromagnetic interaction: spins want to anti-align.*

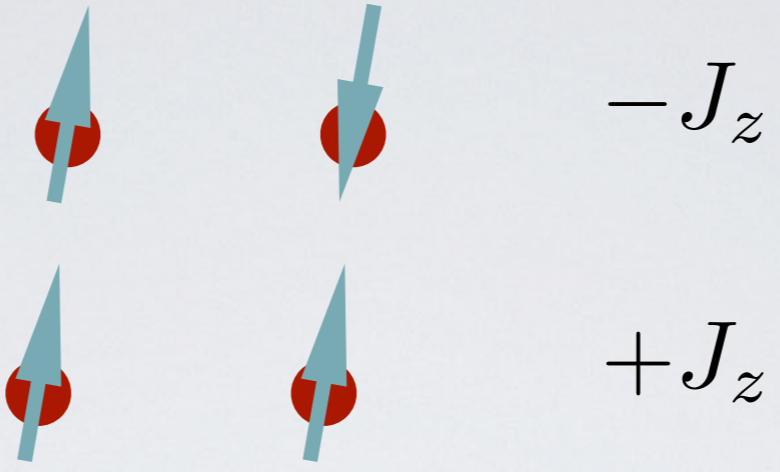
This is the exact wave-function in the Ising limit.

This phase survives down to  $J_z = 1$

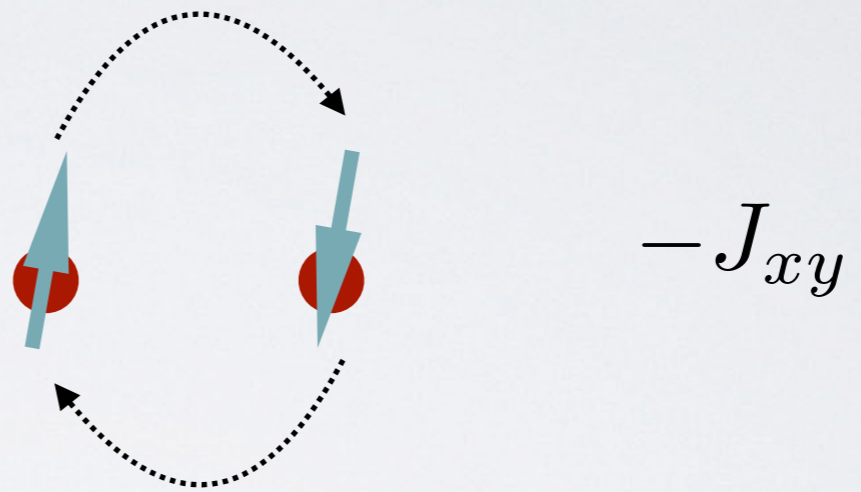
Bipartite lattices are all understood.



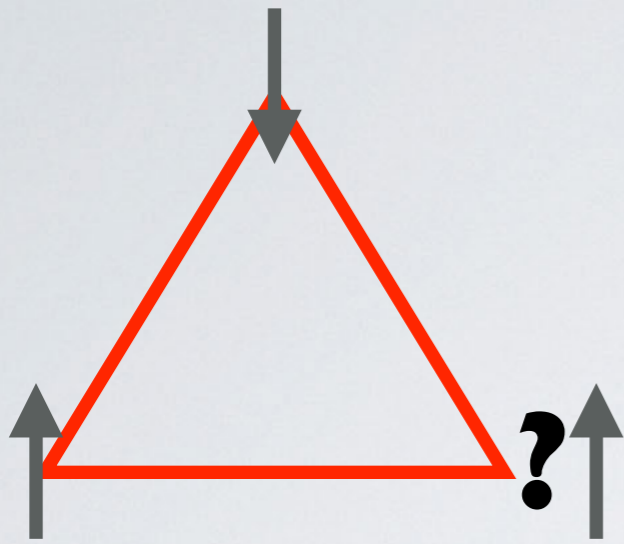
$$H_{ij}^z = \sigma_i^z \sigma_j^z$$



$$H_{ij}^{xy} = (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y)$$



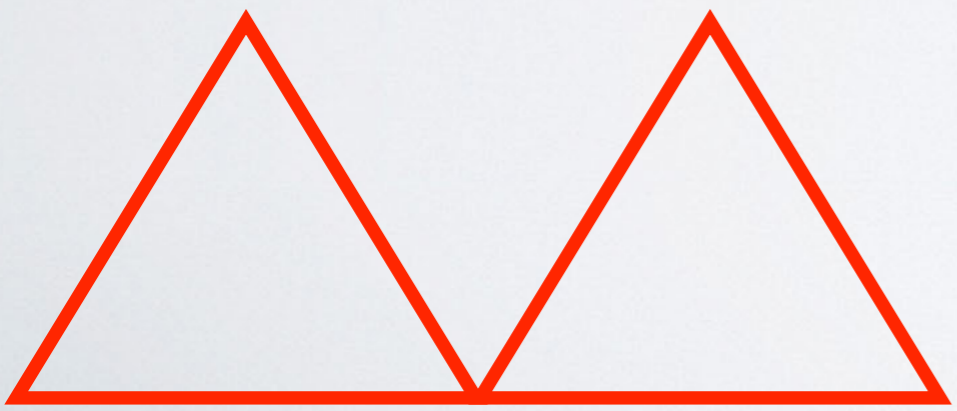
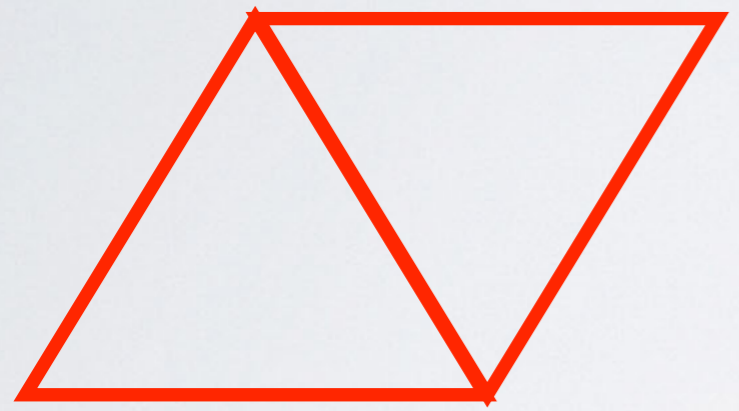
# The story of frustration.



$$H_{xxz} = H_{xy} + J_z \sum_{ij} S_i^z S_j^z \quad (\text{ising limit } J_z \rightarrow \infty)$$
$$H_{\text{ising}} = \sum_{ij} S_i^z S_j^z$$

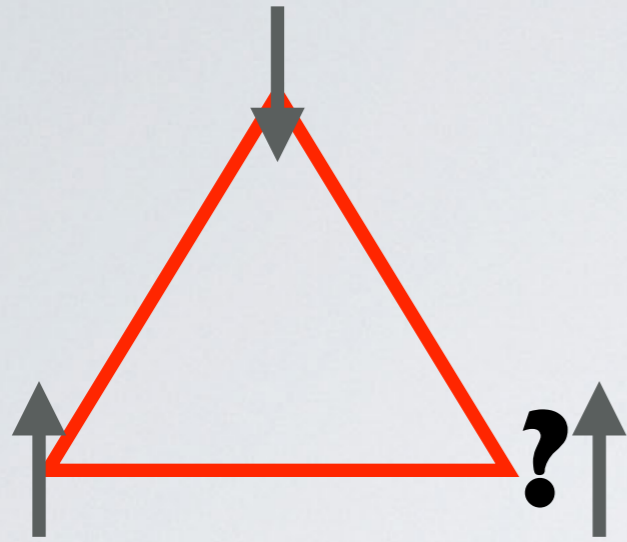


In this Ising limit, however you paste together triangles, there are many degenerate states

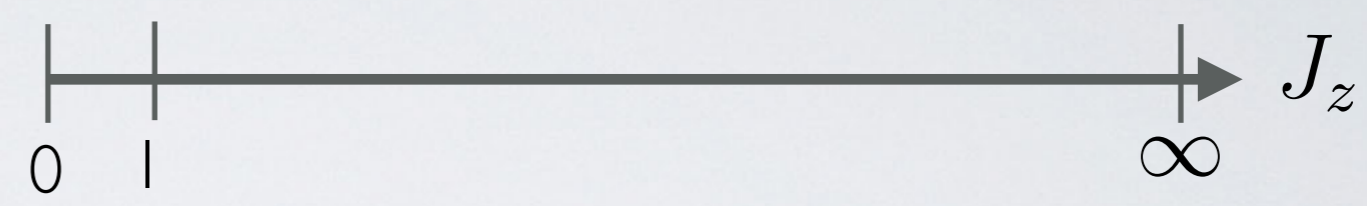


Frustrated magnets!

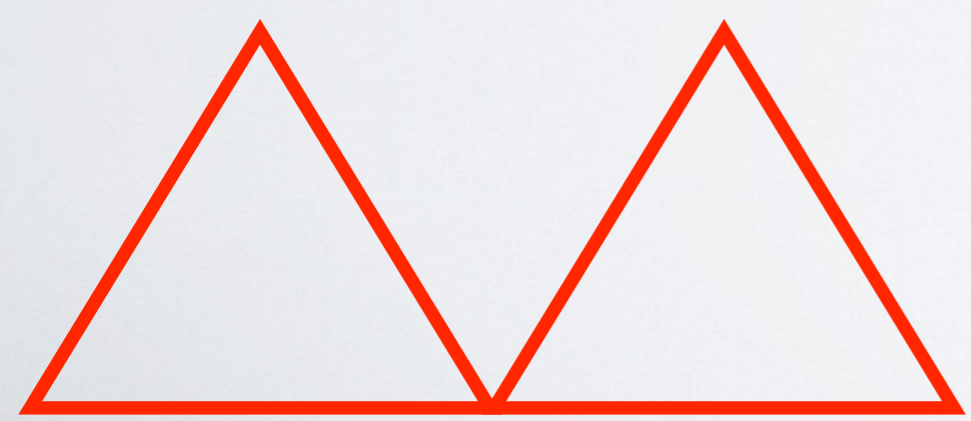
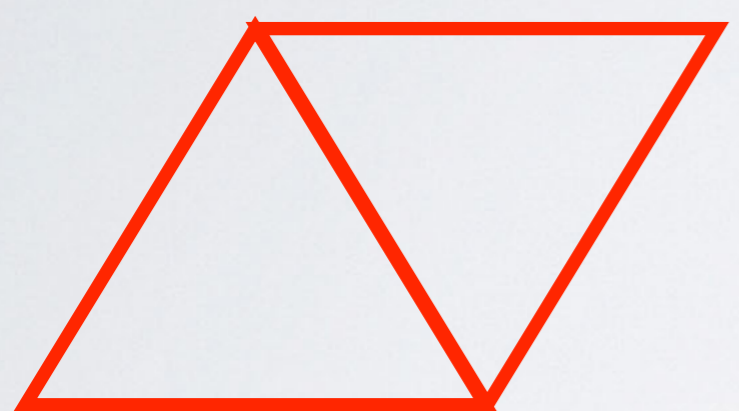
# The **old** story of frustration.



$$H_{xxz} = H_{xy} + J_z \sum_{ij} S_i^z S_j^z \quad (\text{ising limit } J_z \rightarrow \infty)$$
$$H_{\text{ising}} = \sum_{ij} S_i^z S_j^z$$



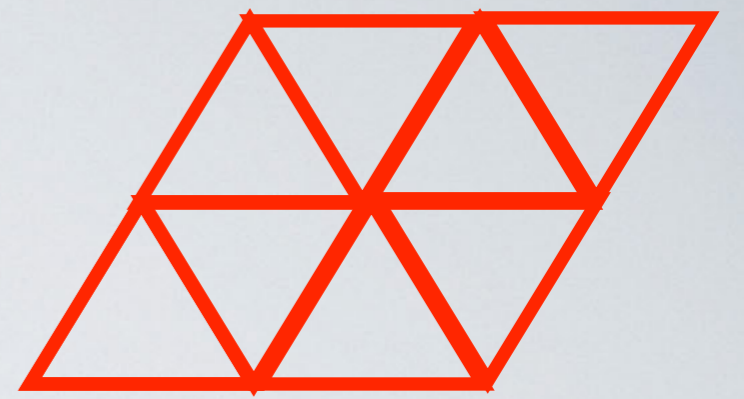
In this Ising limit, however you paste together triangles, there are many degenerate states



Frustrated magnets!



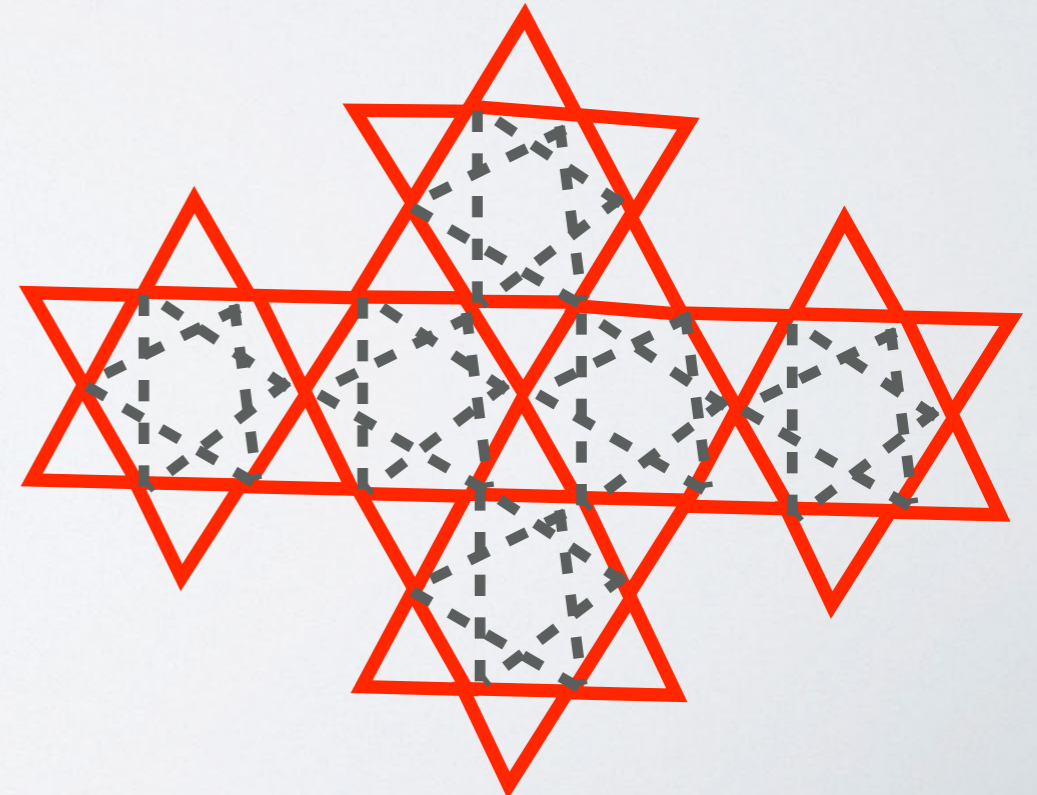
**Triangle lattice**



**Kagome Lattice**



**$J_1$ - $J_2$  Kagome Lattice**

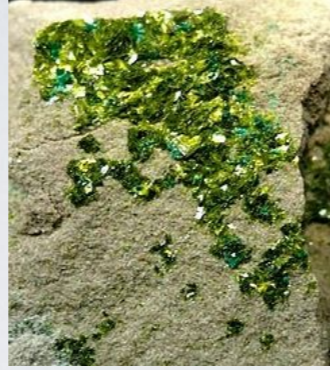




# Kagome Materials



**Herbertsmithite**



**Volborthite**



**Kapellasite**



**Vesigniette**

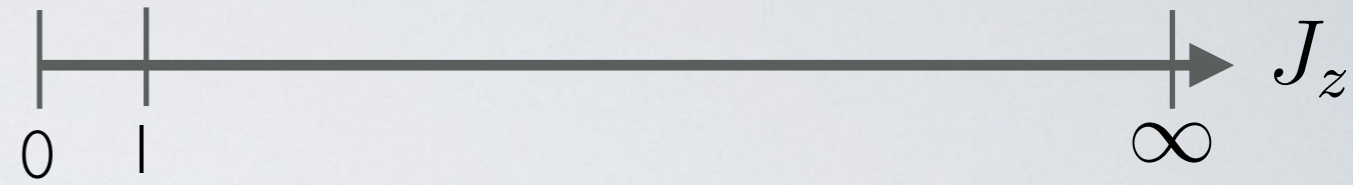
# Phil Anderson suggested that the n.n. Heisenberg model on the triangular lattice wasn't a neel state (**frustration!**)



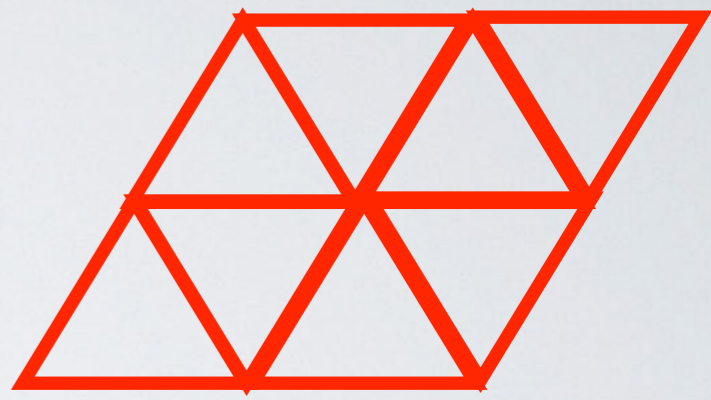
RESONATING VALENCE BONDS: A NEW KIND OF INSULATOR?\*

P. W. Anderson  
Bell Laboratories, Murray Hill, New Jersey 07974  
and  
Cavendish Laboratory, Cambridge, England

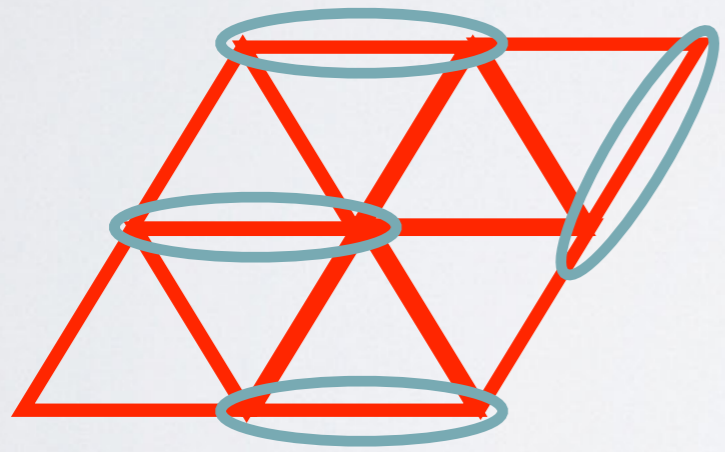
(Received December 5, 1972; Invited\*\*)



**instead, he suggested it was a RVB state.  
(today we would call such a thing a spin-liquid).**



$$|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$$

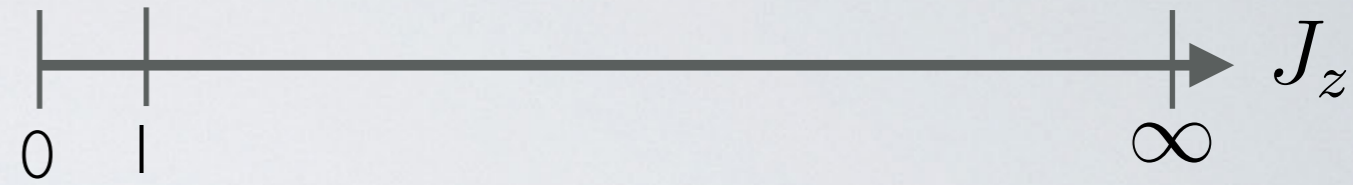




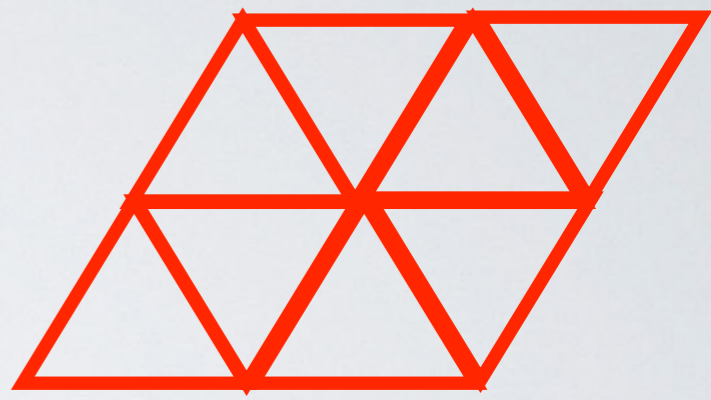
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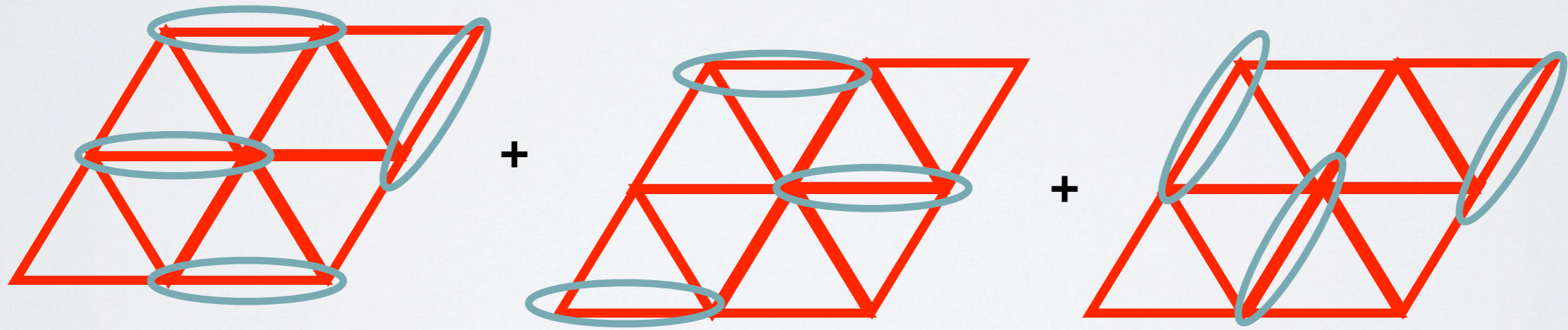
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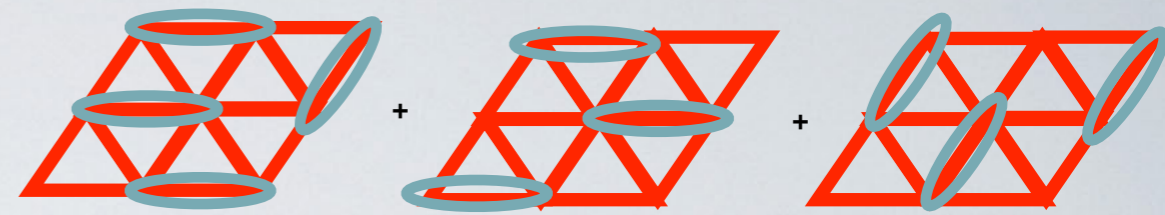


$$|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$$



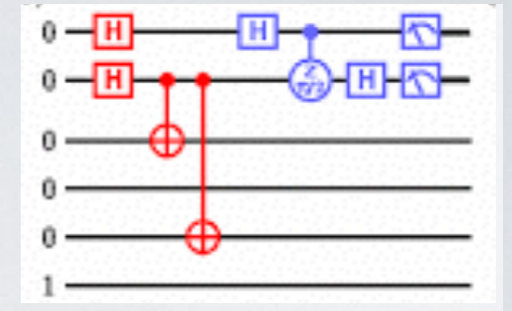
Like Benzene

The hunt for **spin liquids** is one of the forefront areas of condensed matter research!

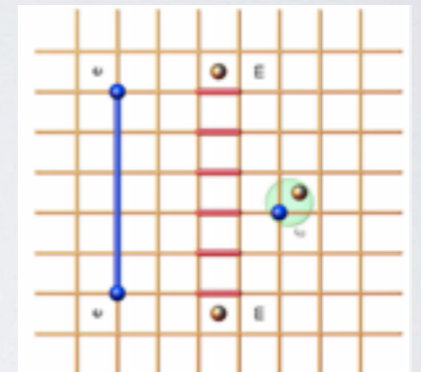


*Beyond the Landau theory of phases* - no broken symmetries!

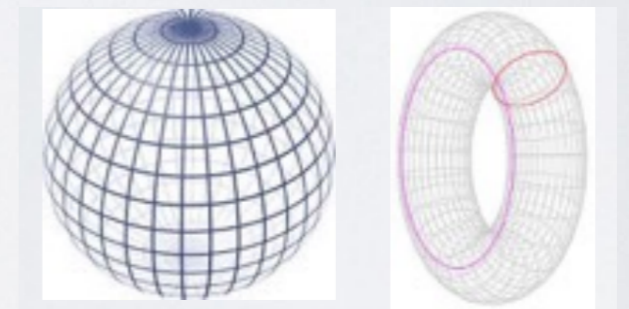
*Long Range Entanglement* - Can't be produced from a product state via a short quantum circuit



*Fractionalized Excitations* - Electron breaks into multiple emergent pieces.



*Topological Degeneracy* - Manifold dependent geometry



The search for spin liquids is truly a hunt. We haven't had any good story for what sort of lattices should support spin liquids.



Was Phil Anderson right? Was the triangular lattice at the Heisenberg point a spin-liquid?

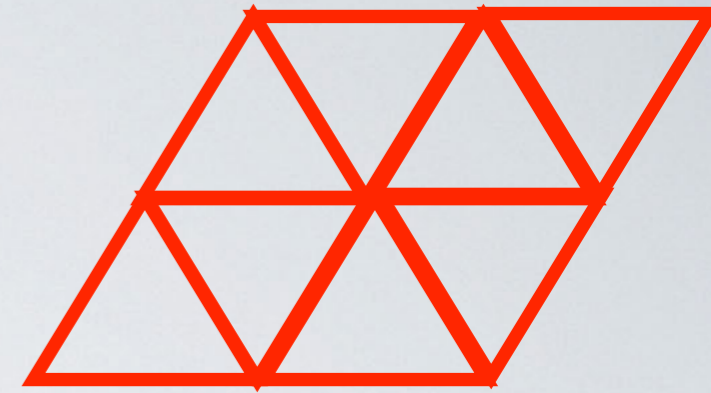
No!

How do we know?

Years and years of numerics...

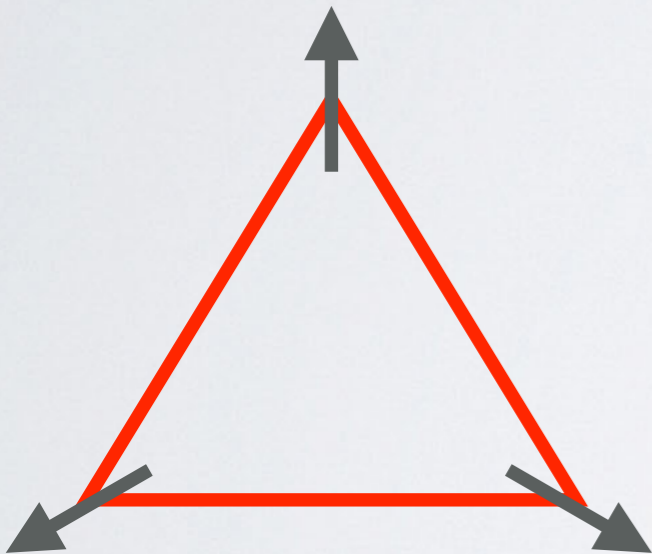
Huse-Elser

Green's Function Monte Carlo

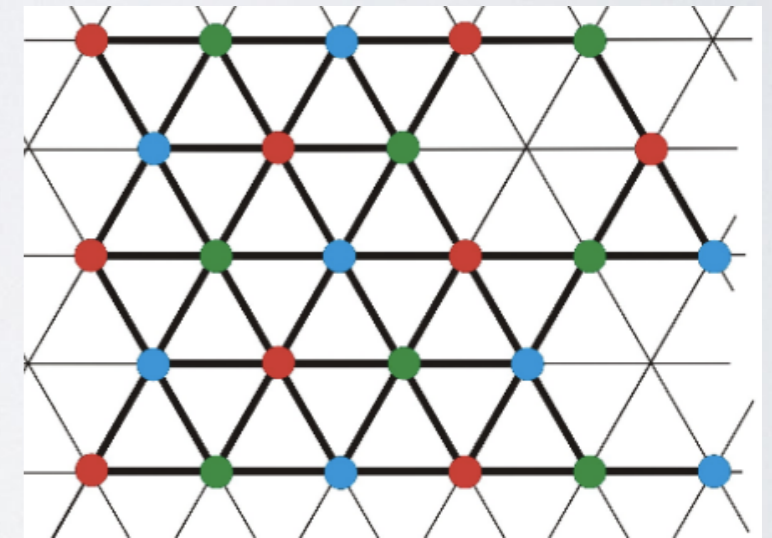


The triangular lattice is 120 degree ordered.

No hint from the frustrated Ising limit.



Define  
3 "colors"

$$|a\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle) \quad \bullet$$
$$|b\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + \omega|\downarrow\rangle) \quad \bullet$$
$$|c\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + \omega^2|\downarrow\rangle) \quad \bullet$$
A diagram of a triangle with three colored vertices: red, blue, and green. The lines are red.

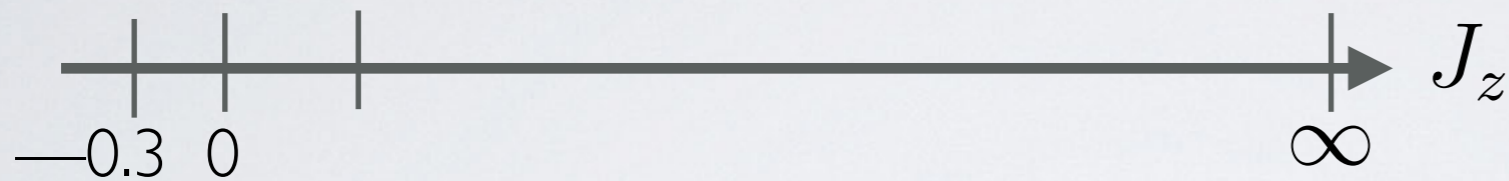
*Morally, but not actually the wave-function. The actual wave-function is a highly complicated dressed version of this (that's why it took forever to verify this).*

**Why?** Because numerics says so....

How about if we have negative  $J_z$ ?

$$H_{xxz} = H_{xy} + J_z \sum_{ij} S_i^z S_j^z \quad (\text{ising limit } J_z \rightarrow \infty)$$

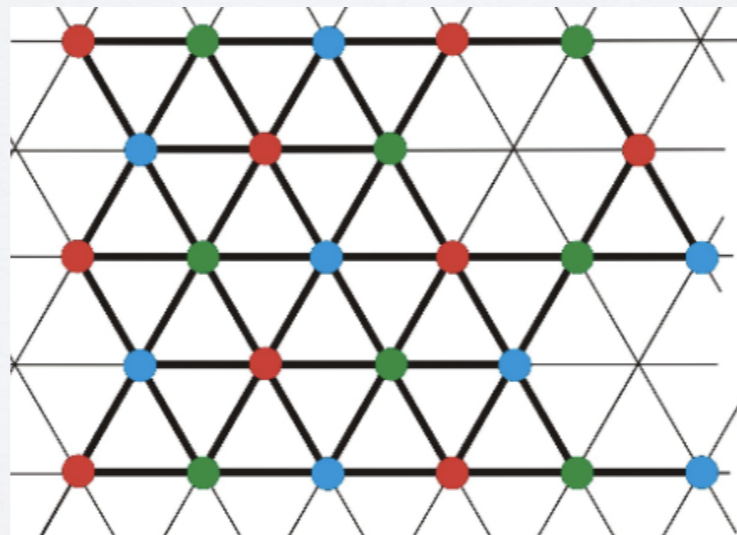
$$H_{\text{ising}} = \sum_{ij} S_i^z S_j^z$$



Wouldn't expect a spin-liquid.  
But at least Ising-like?

**Nope...**

coplanar



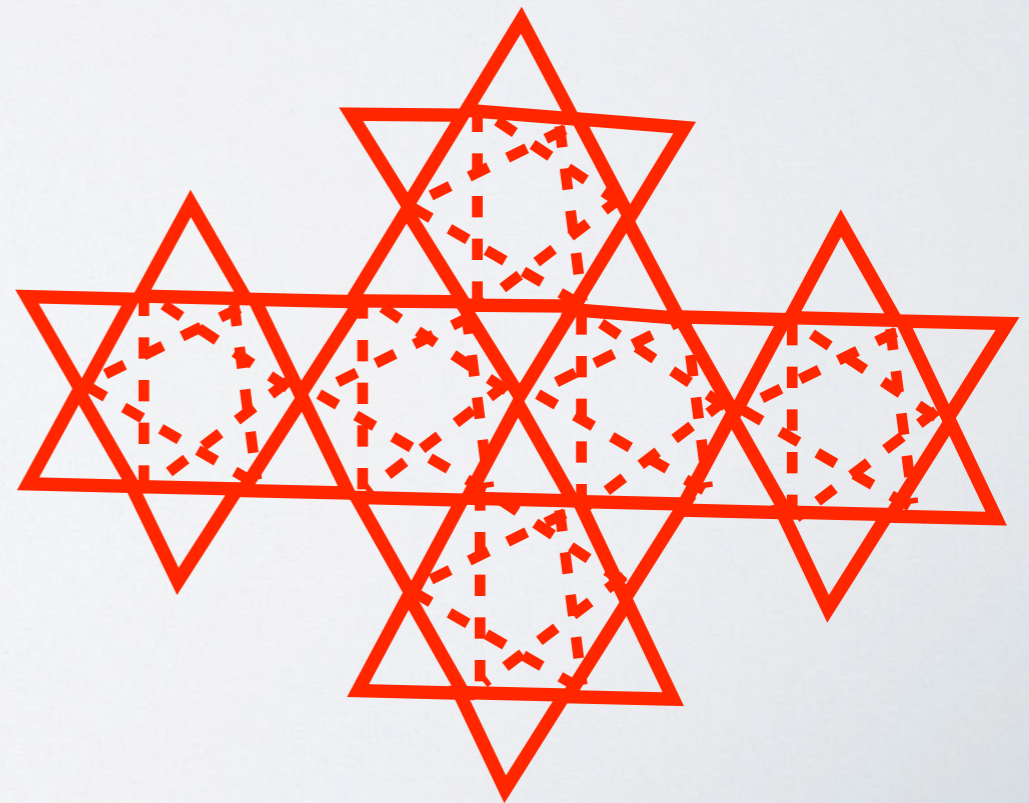


What about the J1-J2 kagome lattice?

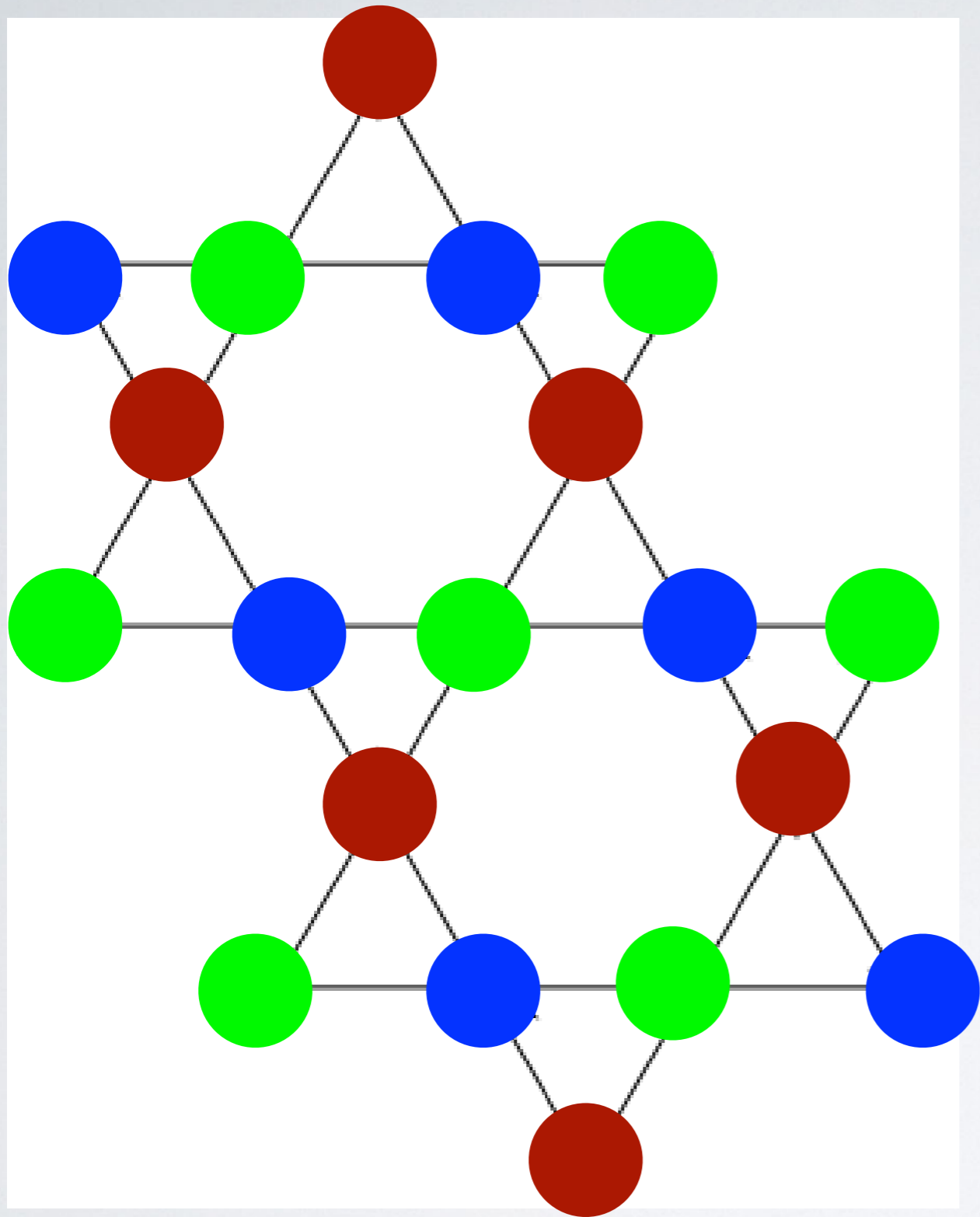
$$J_1 = 1$$

$$J_2 = 1$$

$$J_z = 1$$



What about the J1-J2 kagome lattice?

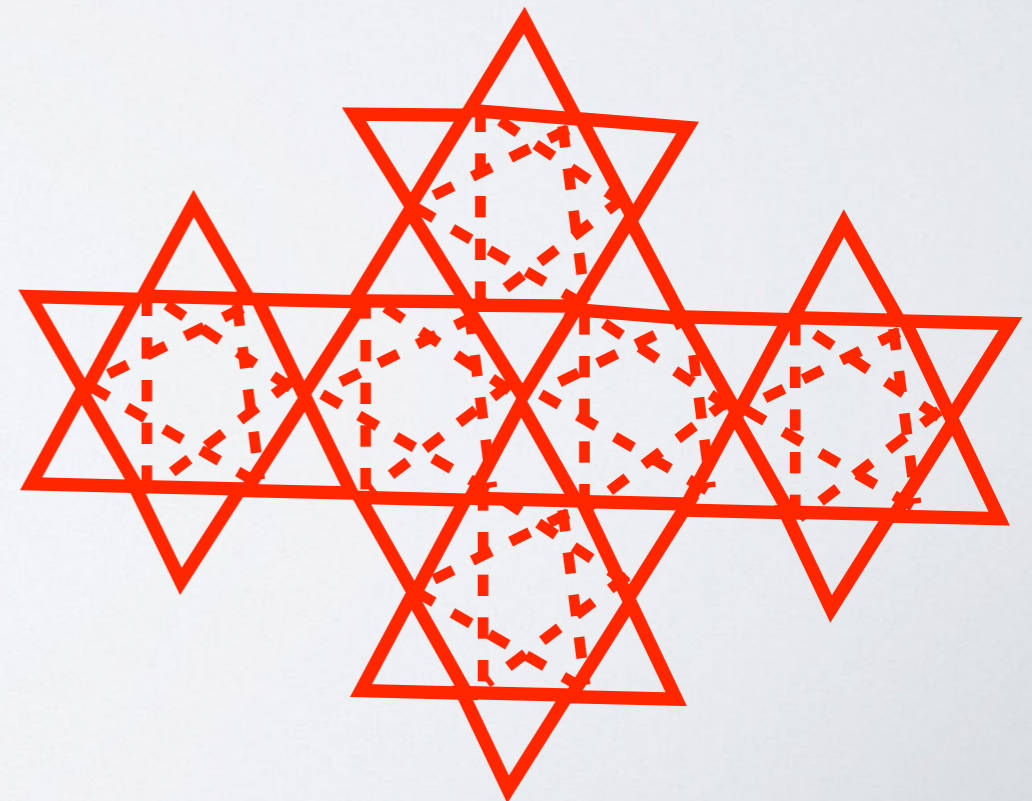


coplanar ( $q=0$ )

$$J_1 = 1$$

$$J_2 = 1$$

$$J_z = 1$$

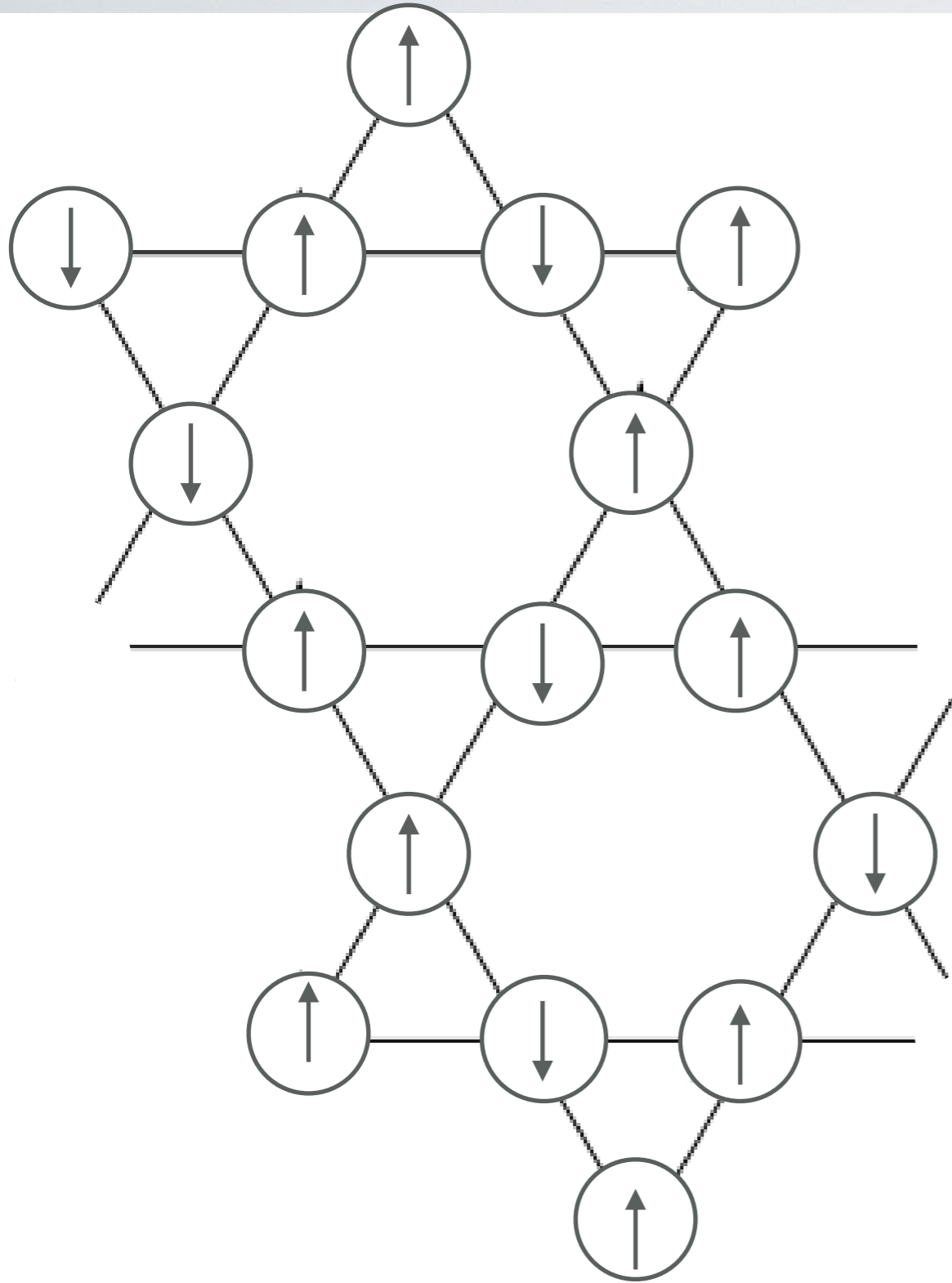


**Why?** Because numerics says so....



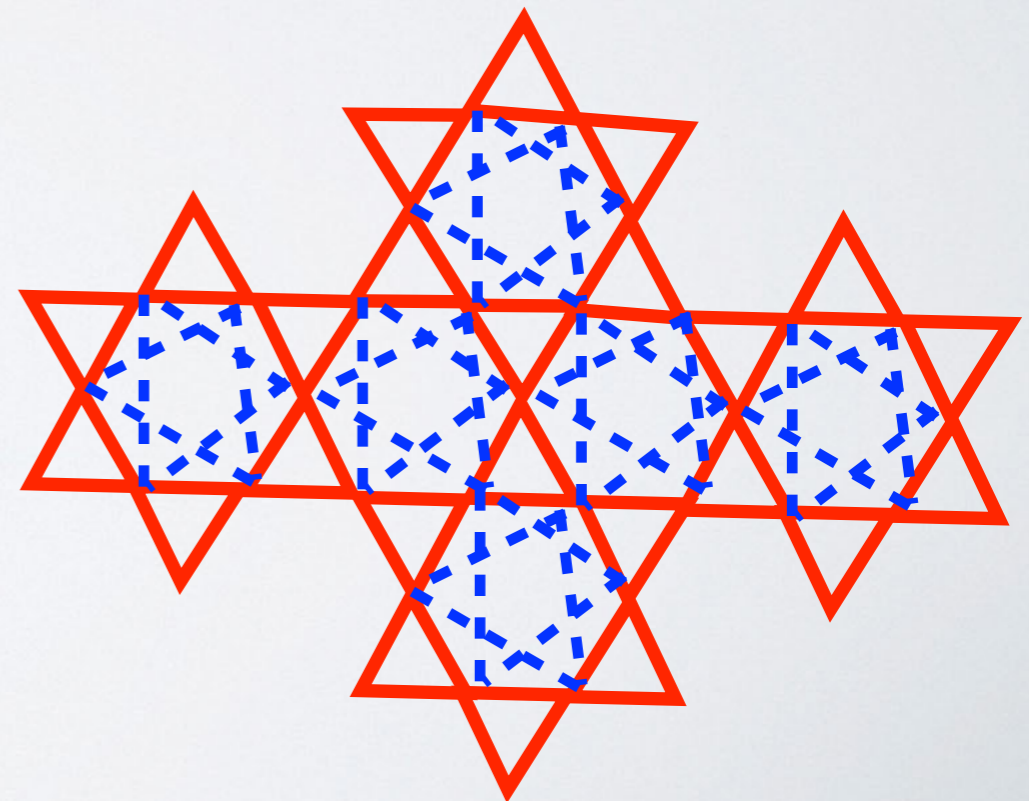
What about the J1-J2 kagome lattice?

$$\begin{aligned} J_1 &= 1 \\ J_2 &= -1 \\ J_z &= \infty \end{aligned}$$



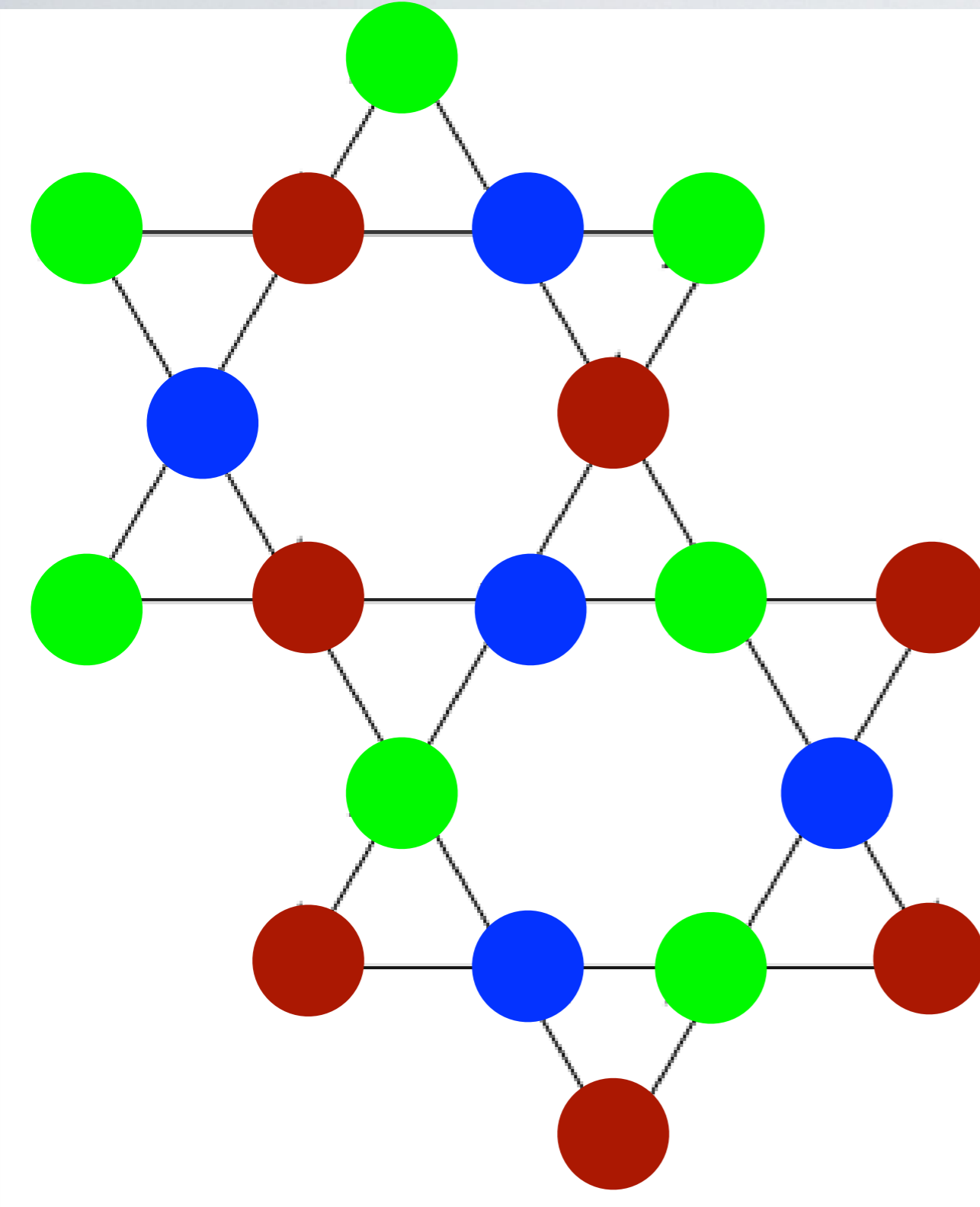
This is the answer in the Ising Limit

Surely, it must be this at  $J_z = 1$



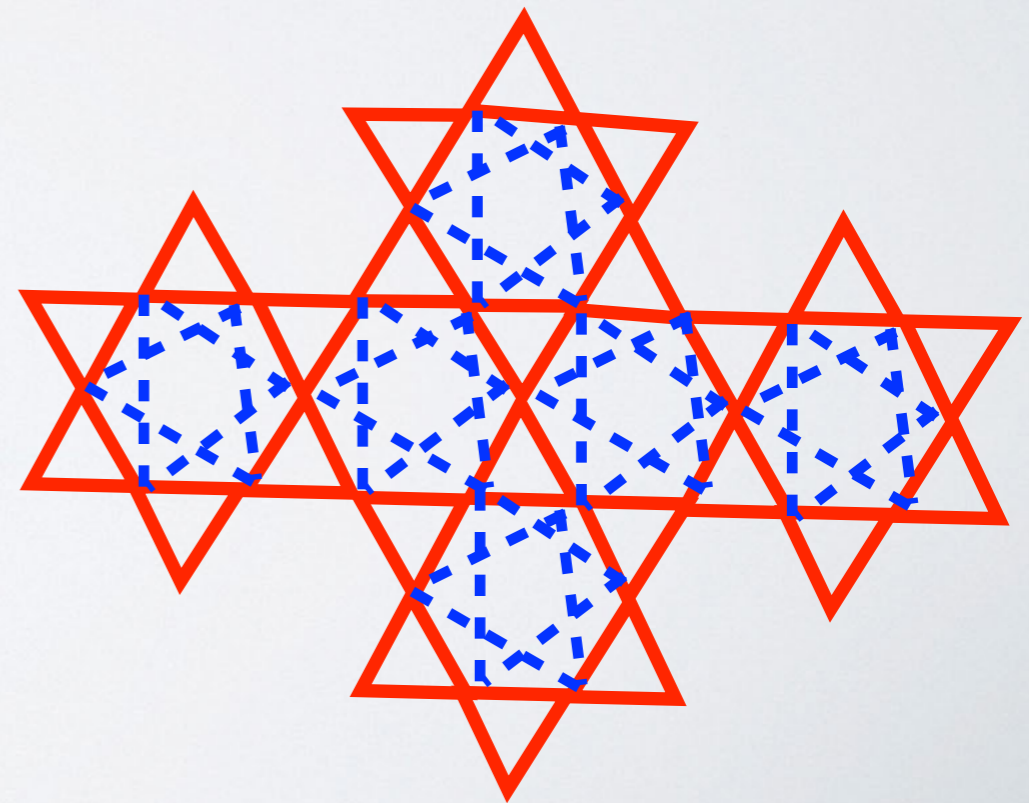
What about the J1-J2 kagome lattice?

$$J_1 = 1$$
$$J_2 = -1$$
$$J_z = 1$$



coplanar  $\sqrt{3} \times \sqrt{3}$

**Why?** Because numerics says so....





# Is everything coplanar?

Triangular:  $J_z = 1$  coplanar

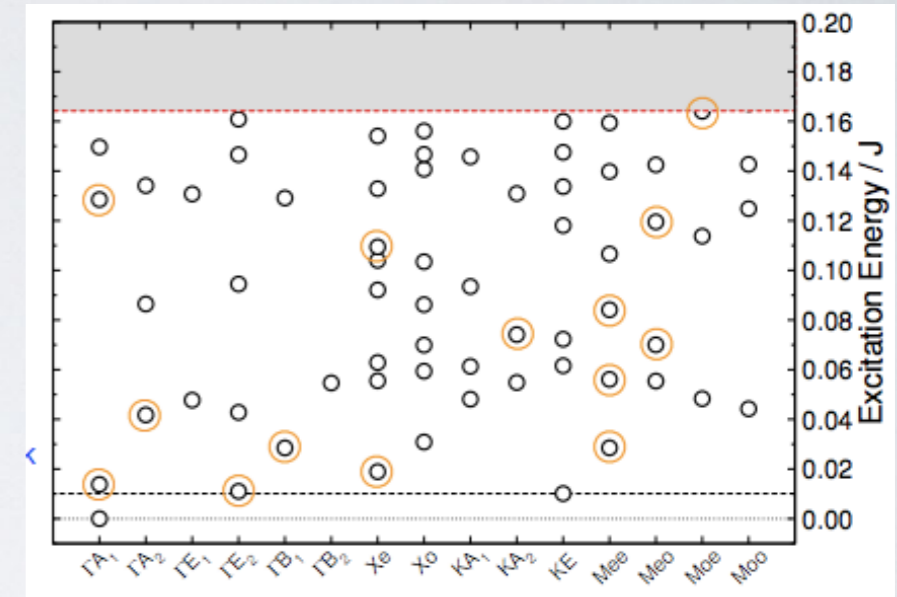
$J_z = -0.3$  coplanar

Kagome:  $J_2 = 1$   $J_z = 1$  coplanar

$J_2 = -1$   $J_z = 1$  coplanar

$J_2 = 0.5$   $J_z = 1$  Spin Liquid! (White/Huse)  
but a low energy mess

**Why?** years and years of numerics



Lauchli, et. Al

# Kagame spin liquids everywhere....

**Z2 (or Dirac) Spin Liquid**

Heisenberg (White/Huse)

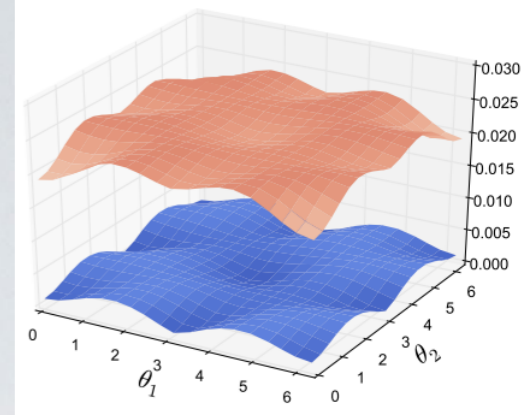
**Chiral Spin Liquid**

2/3 Plateau (this work)

1/3 Plateau (Donna Sheng)

Chiral Term (Bela Bauer, Andreas Ludwig)

$J_1, J_2, J_3$  (Donna Sheng)





# Experimental evidence for spin-liquids in hyperkagome

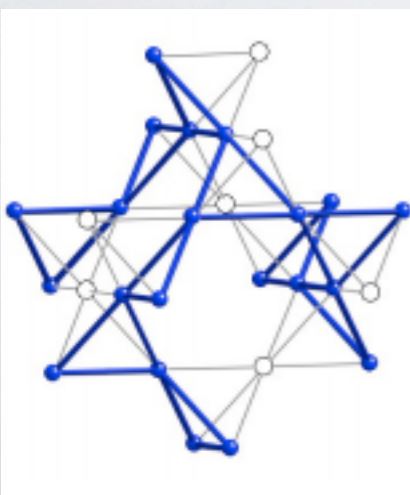
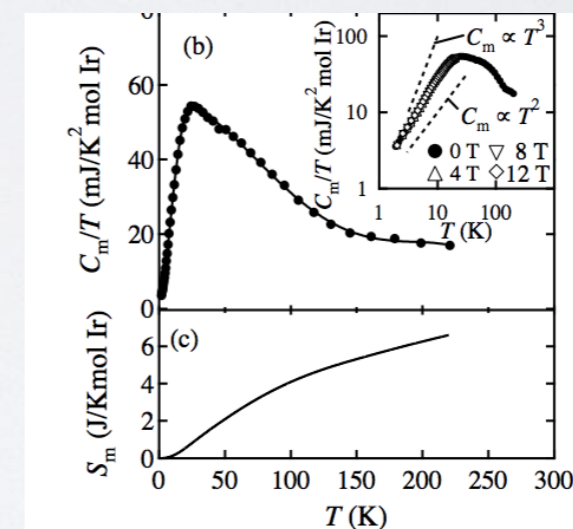
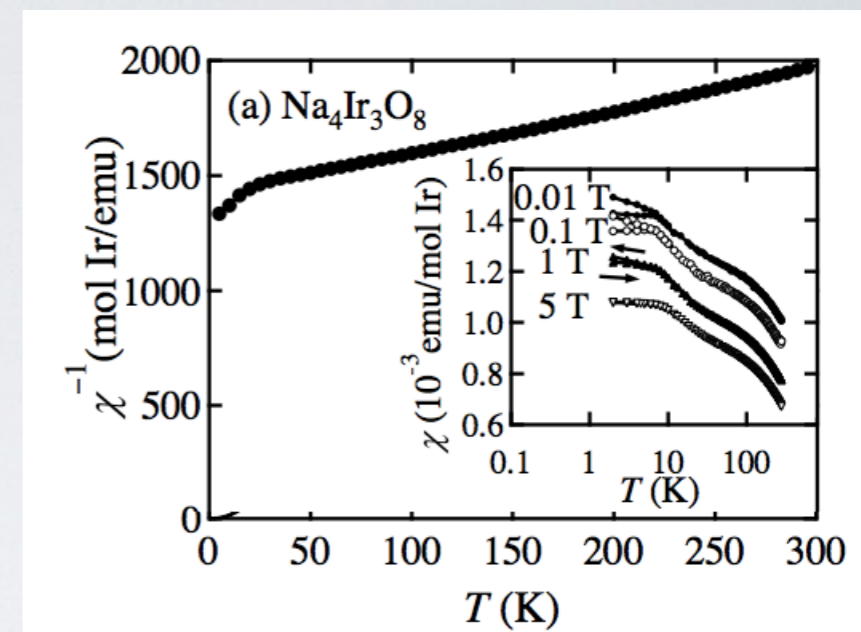


(depleted pyrochlore)

No sign of magnetic ordering down to a few Kelvin

Curie-Weiss temperature of 650K

Gapless excitations



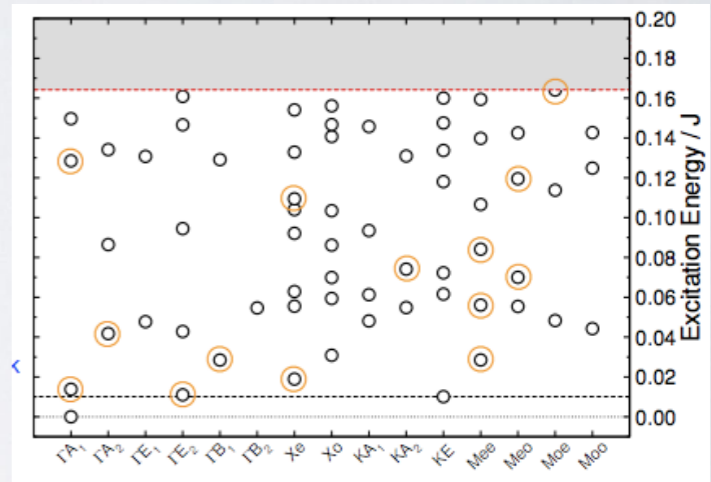
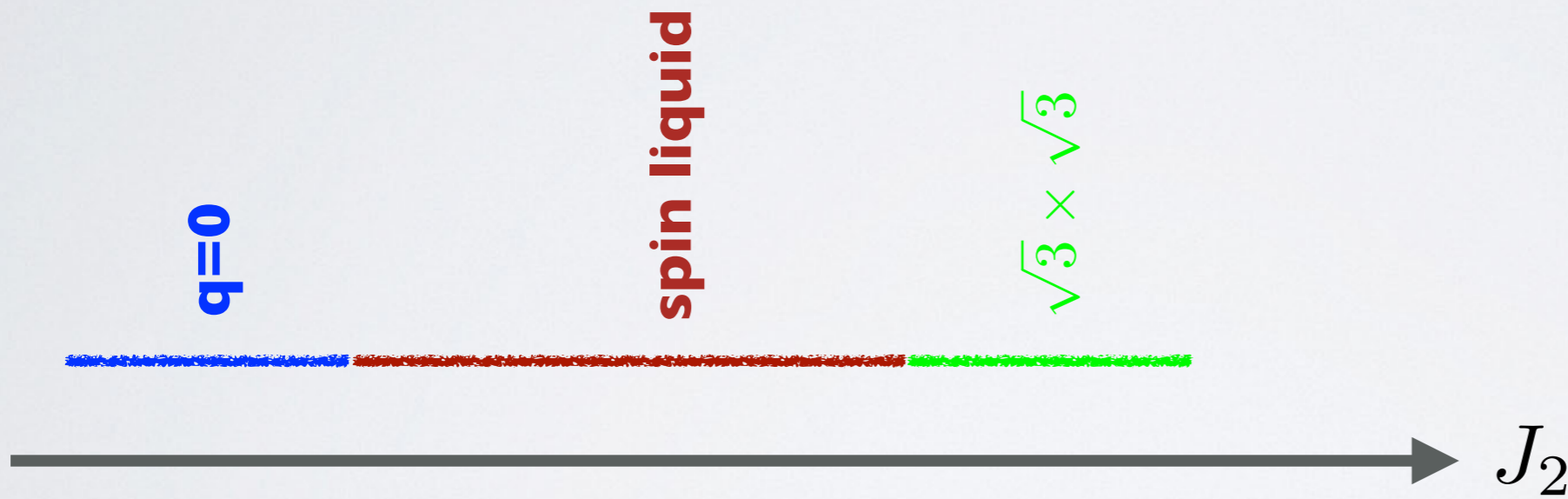
Y. Okamoto, M. Nohara, H. Aruga-Katori, and H. Takagi  
(2006), arXiv:0705.2821

# The Ising frustration doesn't seem to be a good explanation for anything.

(1) Why hyperkagome and kagome and not triangular?

Both are equally frustrated in the ising limit.

(2) Ising seems to have little to do with the coplanar phases



(3) Why does kagome have so many real phases and so many competing low-energy states?



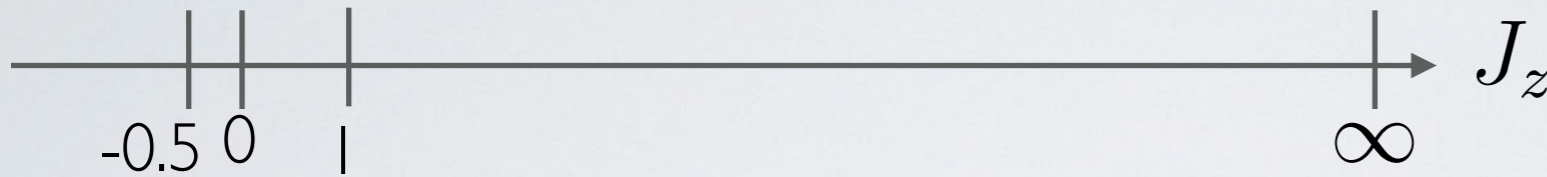
**Is the take away point that strongly correlated systems is hard.**

So... use numerics :)

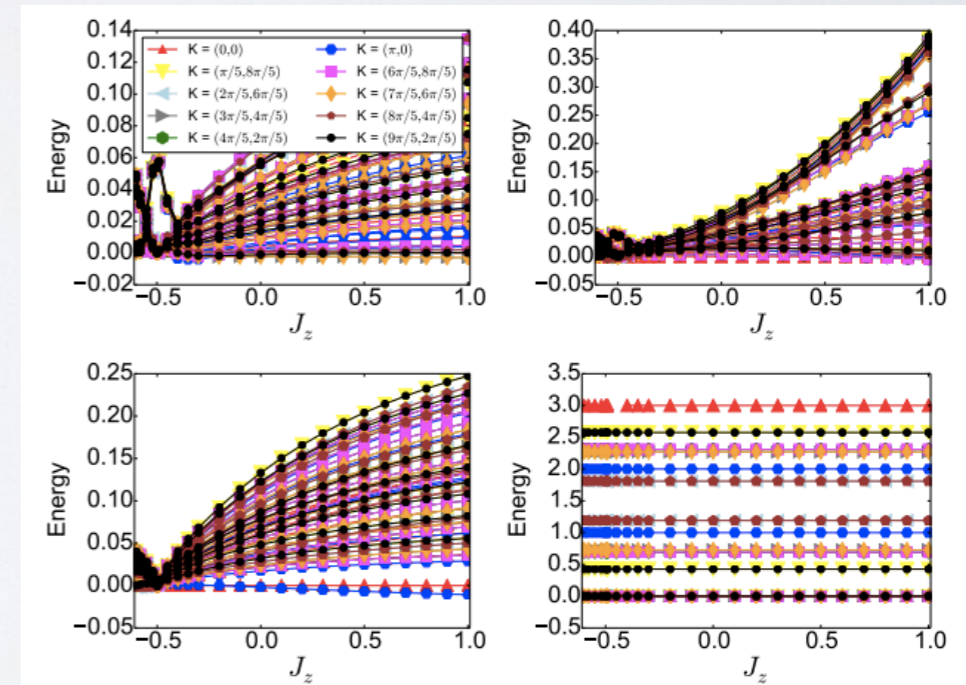
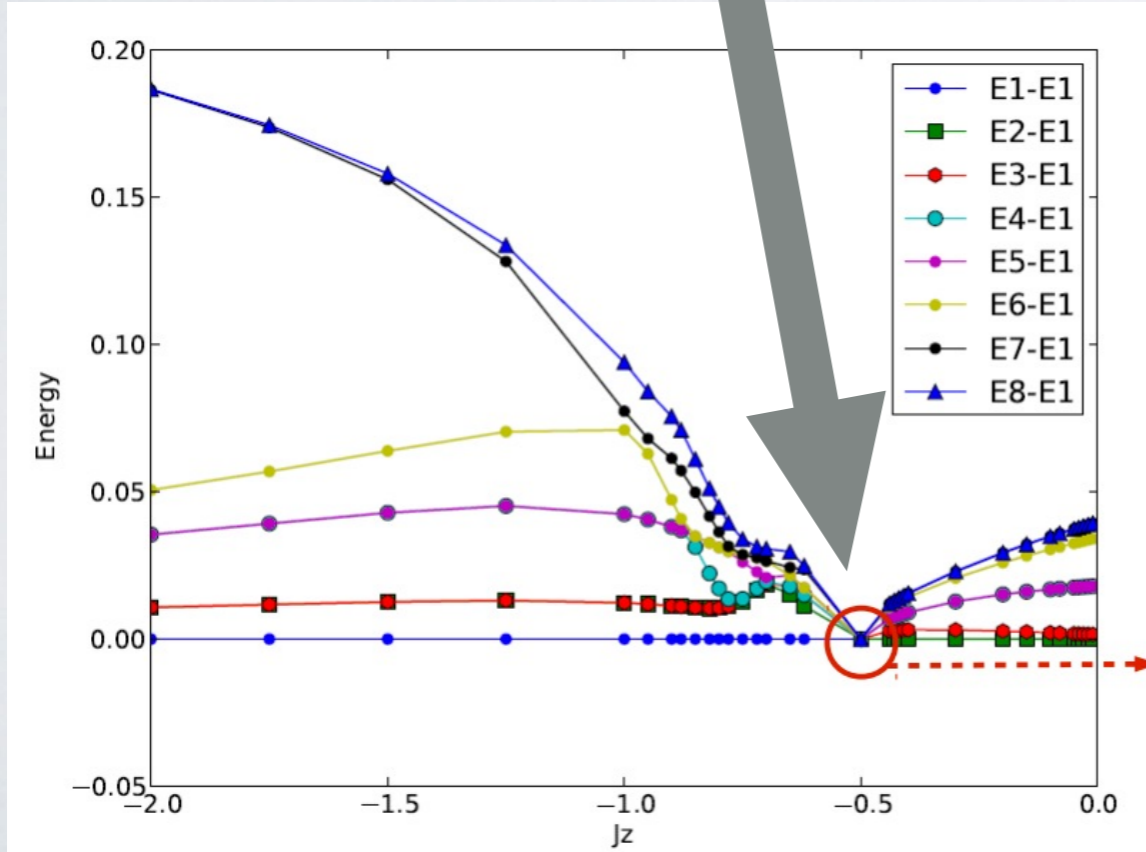
Q: Is there a better story?

# An interesting discovery.... (amazing it hasn't been known for 30 years)

$$H = \sum_{ij} S_i^x S_j^x + S_i^y S_j^y - 0.5 \sum_{ij} S_i^z S_j^z$$



**On the kagome:  
massive exact degeneracy in the XXZ model!  
exactly  $-J/4$**

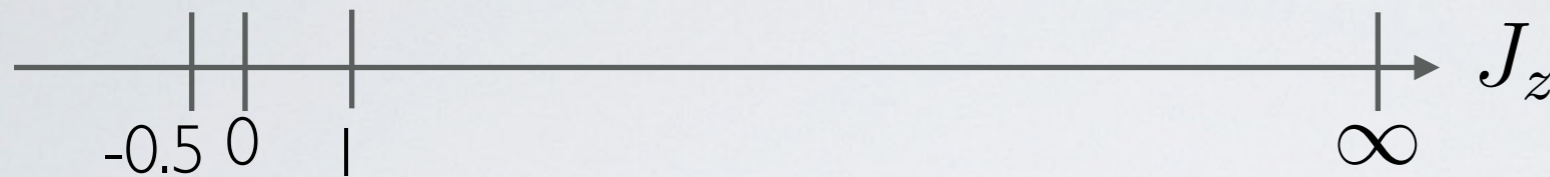


**What's going on? Who ordered this?**



An interesting discovery.... (amazing it hasn't been known for 30 years)

$$H = \sum_{ij} S_i^x S_j^x + S_i^y S_j^y - 0.5 \sum_{ij} S_i^z S_j^z$$

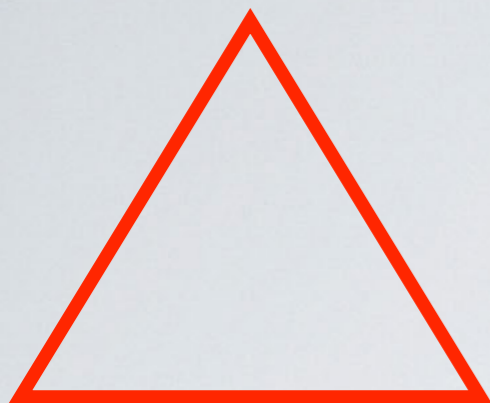


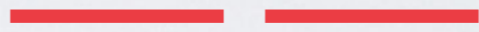
**On the triangle:  
no massive exact degeneracy.  
exactly  $-J/2$**

**What's going on? Who ordered this?**

Who ordered that?

$$H_{XXZ0} = \sum_{ij} S_i^x S_j^x + S_i^y S_j^y - 0.5 \sum_{ij} S_i^z S_j^z$$



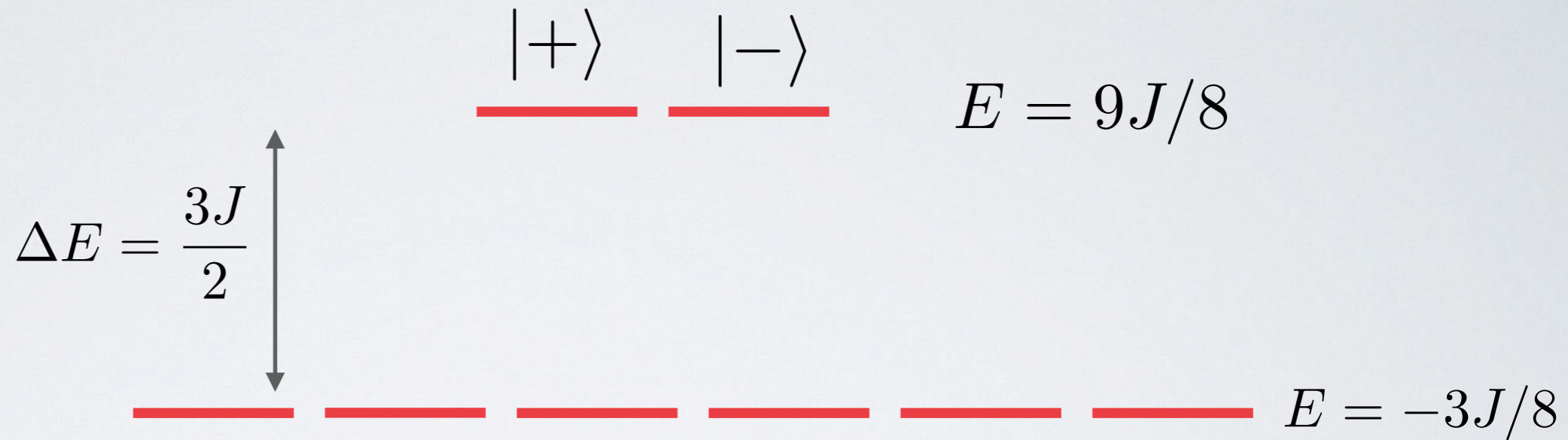
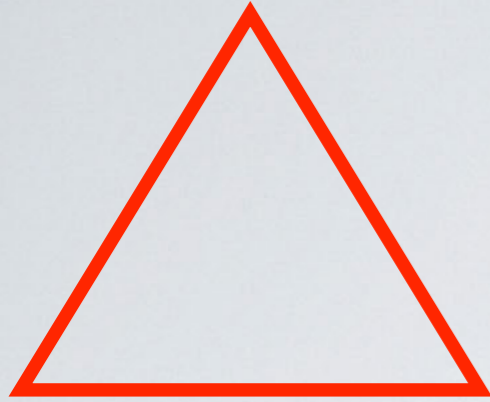

$$E = 9J/8$$


$$E = -3J/8$$



# Who ordered that?

$$H_{XXZ0} = \sum_{ij} S_i^x S_j^x + S_i^y S_j^y - 0.5 \sum_{ij} S_i^z S_j^z$$



$$-\frac{3J}{8} + \frac{3J}{2} (|+\rangle\langle+| + |-\rangle\langle-|)$$

Constant

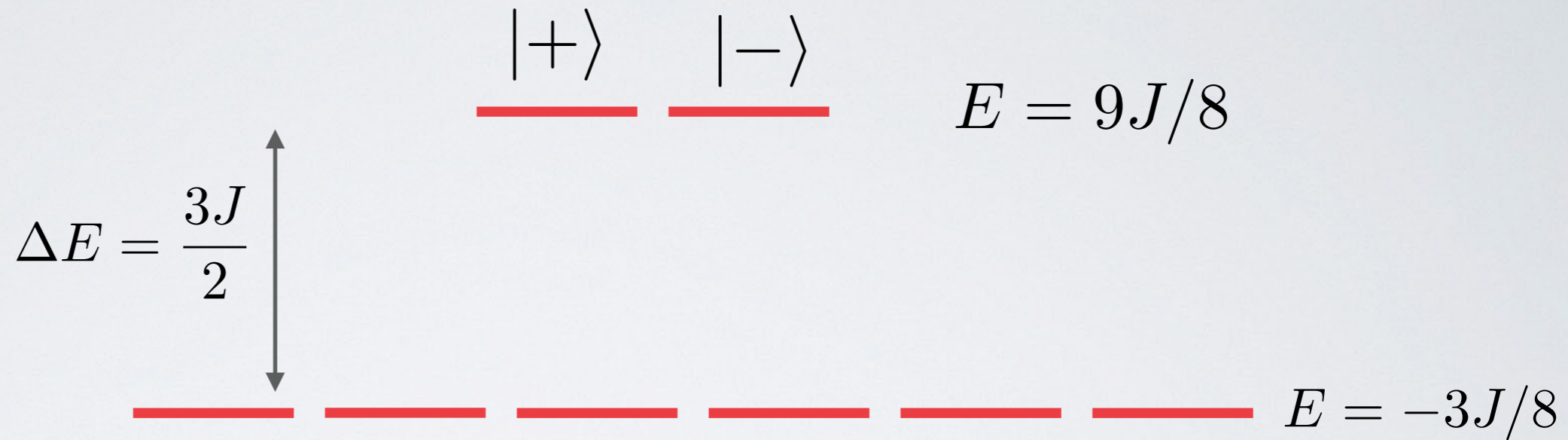
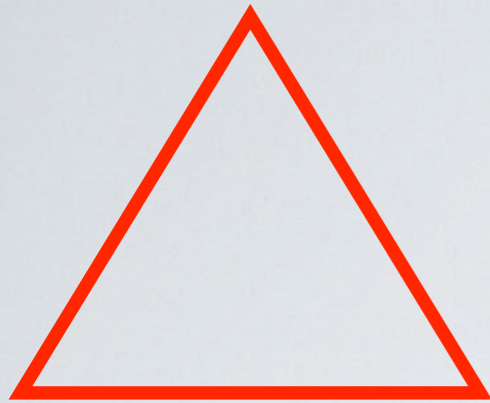
Positive coefficient

Projectors

We want to minimize the energy by zeroing out the projectors

# Who ordered that?

$$H_{XXZ0} = \sum_{ij} S_i^x S_j^x + S_i^y S_j^y - 0.5 \sum_{ij} S_i^z S_j^z$$



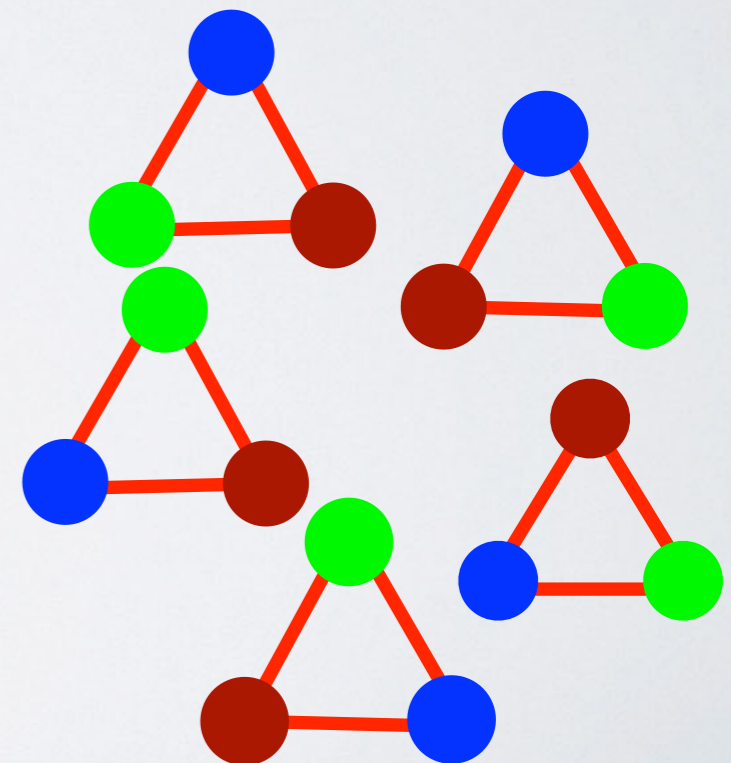
$$-\frac{3J}{8} + \frac{3J}{2} (|+\rangle\langle+| + |-\rangle\langle-|)$$

Constant

Positive coefficient

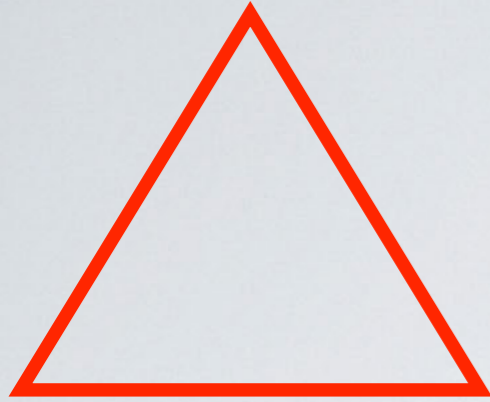
Projectors

We want to minimize the energy by zeroing out the projectors

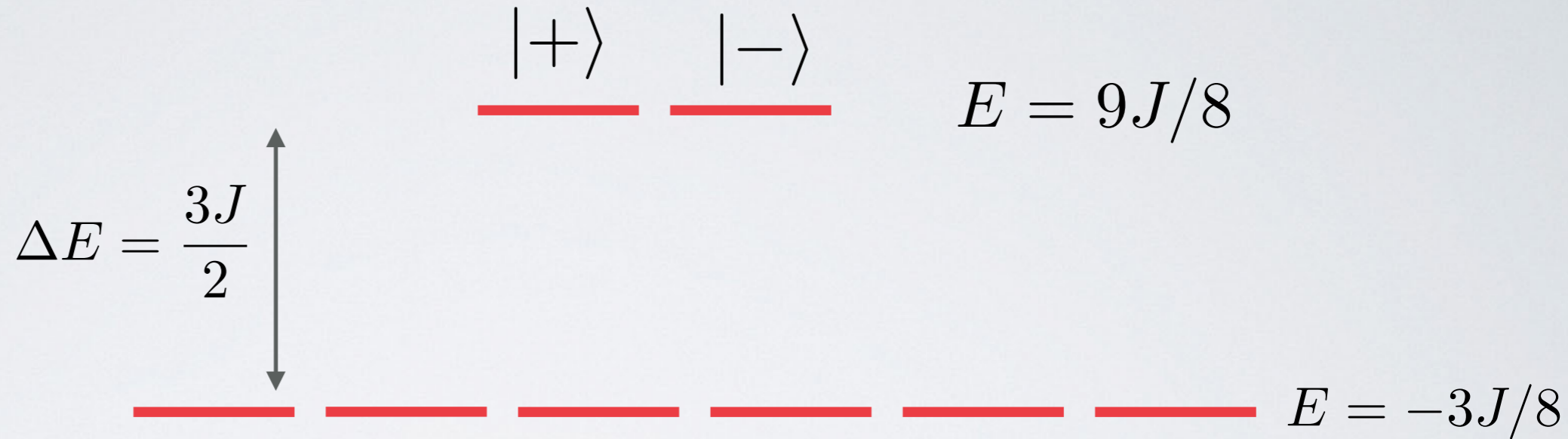




# Who ordered that?



$$H_{XXZ0} = \sum_{ij} S_i^x S_j^x + S_i^y S_j^y - 0.5 \sum_{ij} S_i^z S_j^z$$



$$-\frac{3J}{8} + \frac{3J}{2} (|+\rangle\langle+| + |-\rangle\langle-|)$$

Constant

Positive coefficient

**Projectors**

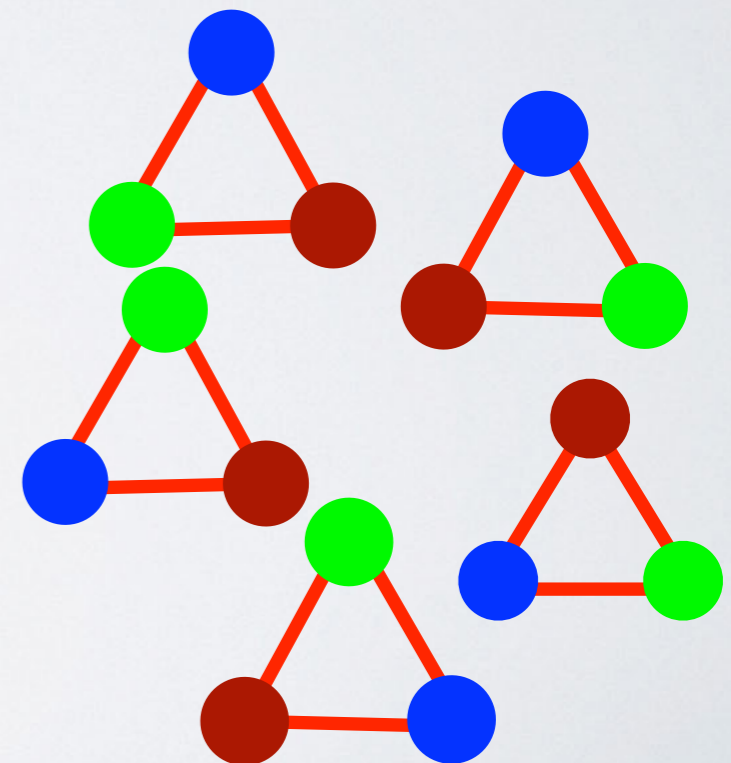
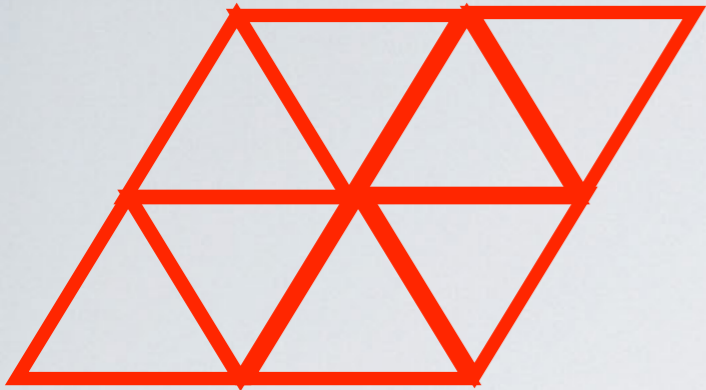
We want to minimize the energy by zeroing out the projectors

- |1⟩ ≡ |↑↑↑⟩
- |2⟩ ≡  $\frac{1}{\sqrt{3}} (|\uparrow\uparrow\downarrow\rangle + \omega|\uparrow\downarrow\uparrow\rangle + \omega^2|\downarrow\uparrow\uparrow\rangle)$
- |3⟩ ≡  $\frac{1}{\sqrt{3}} (|\uparrow\uparrow\downarrow\rangle + \omega^2|\uparrow\downarrow\uparrow\rangle + \omega|\downarrow\uparrow\uparrow\rangle)$
- |4⟩ ≡  $\frac{1}{\sqrt{3}} (|\downarrow\downarrow\uparrow\rangle + \omega|\downarrow\uparrow\downarrow\rangle + \omega^2|\uparrow\downarrow\downarrow\rangle)$
- |5⟩ ≡  $\frac{1}{\sqrt{3}} (|\downarrow\downarrow\uparrow\rangle + \omega^2|\downarrow\uparrow\downarrow\rangle + \omega|\uparrow\downarrow\downarrow\rangle)$
- |6⟩ ≡ |↓↓↓⟩

**What about many triangles?**

$$H = \sum_{\Delta} H_{XXZ0}$$

**Paste together ground states over individual triangles**

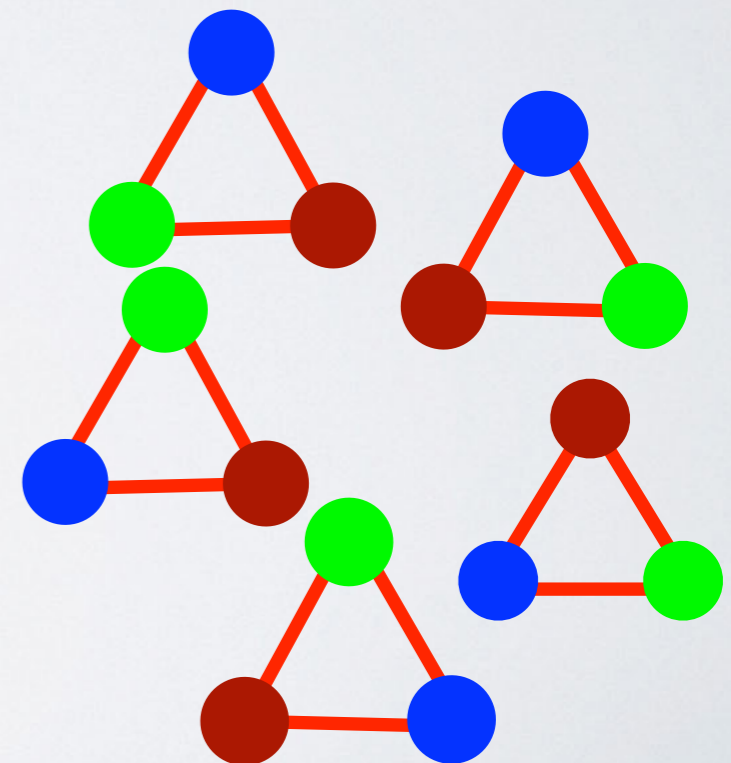
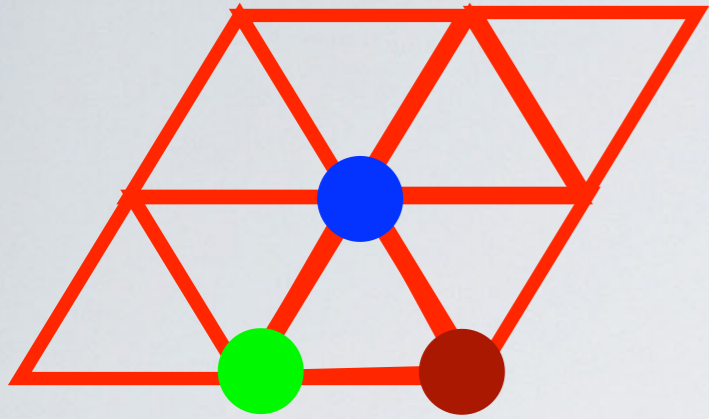




**What about many triangles?**

$$H = \sum_{\Delta} H_{XXZ0}$$

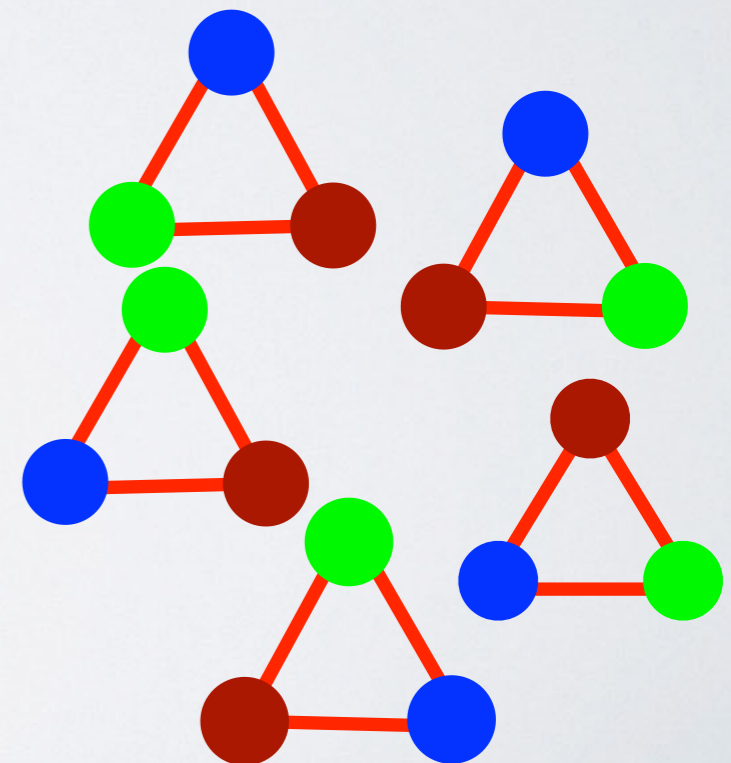
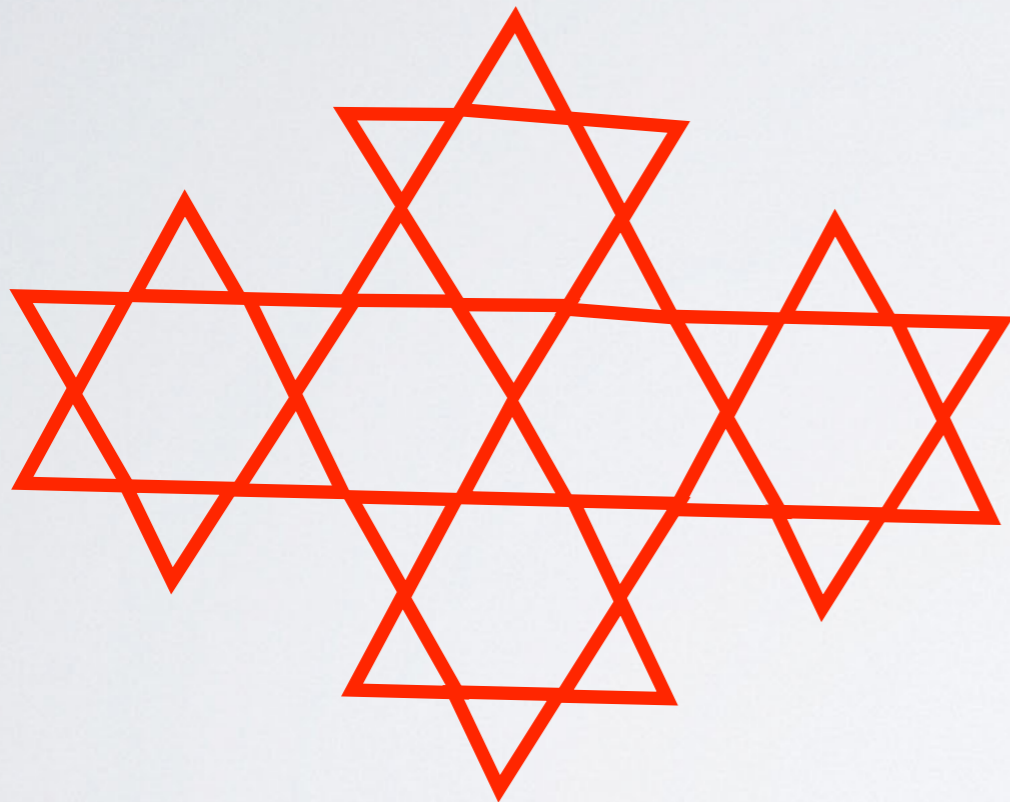
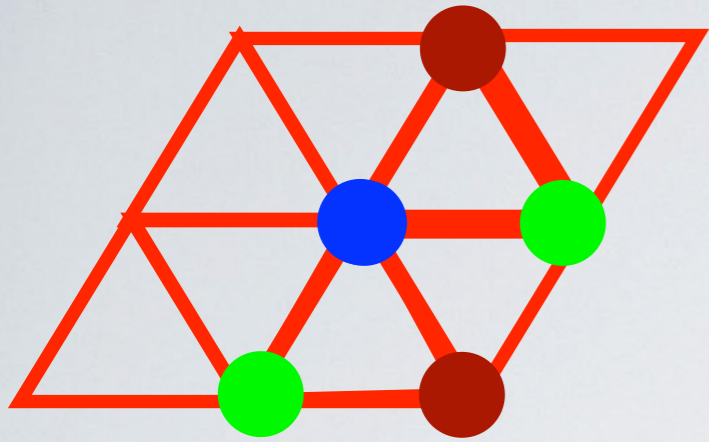
**Paste together ground states over individual triangles**



**What about many triangles?**

$$H = \sum_{\Delta} H_{XXZ0}$$

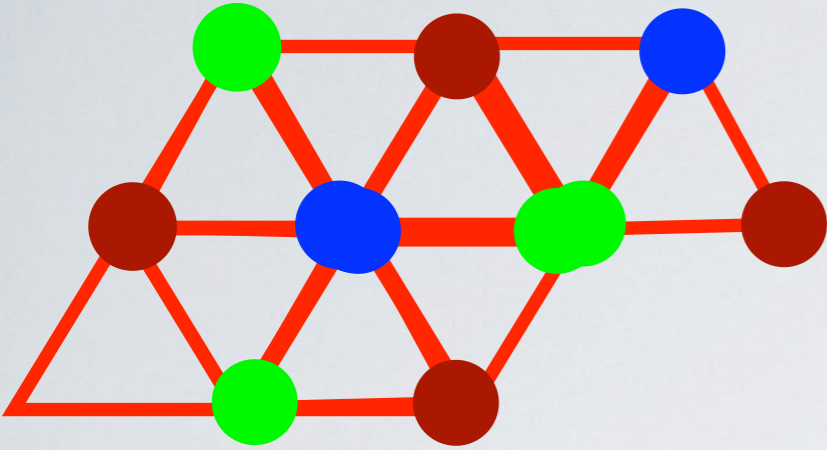
**Paste together ground states over individual triangles**



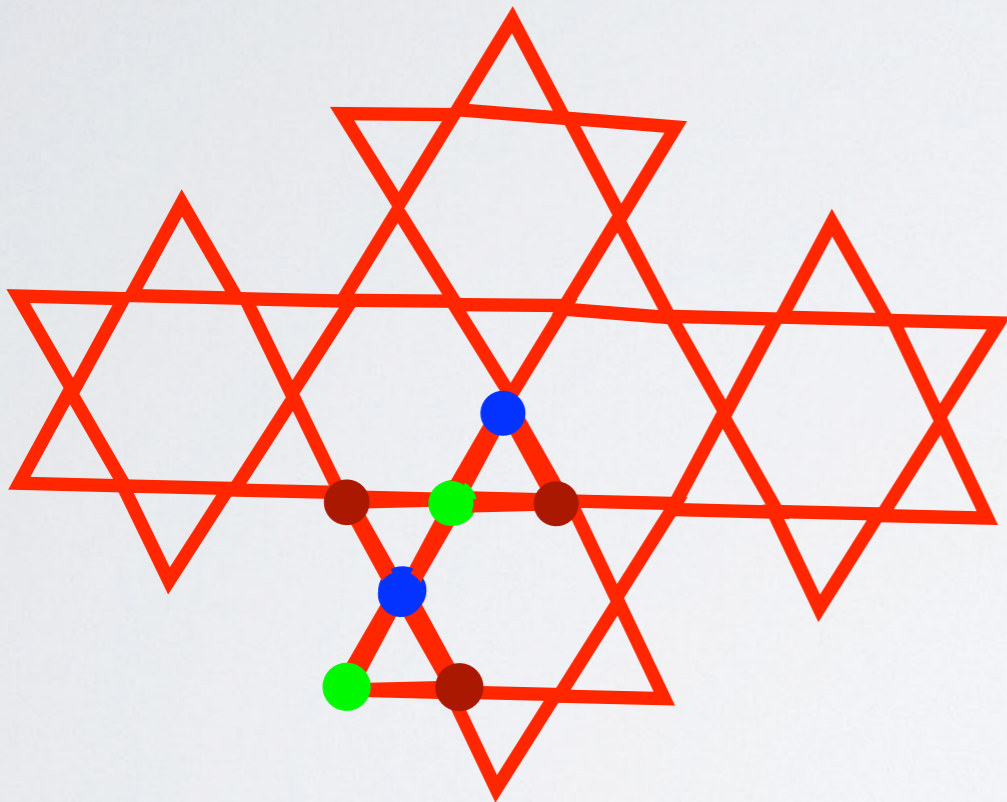


**What about many triangles?**

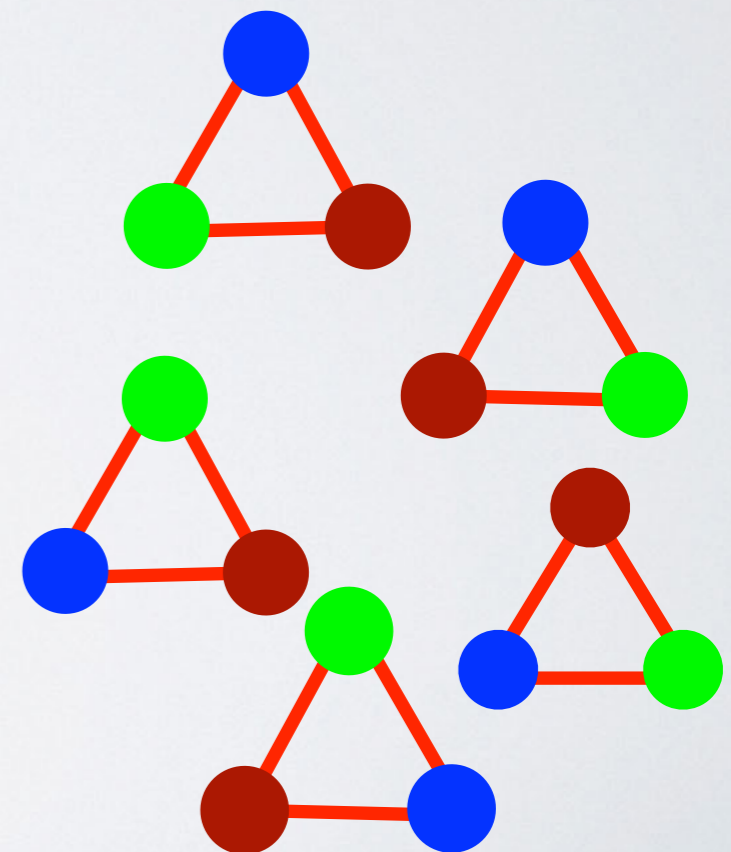
$$H = \sum_{\Delta} H_{XXZ0}$$



**Paste together ground states over individual triangles**

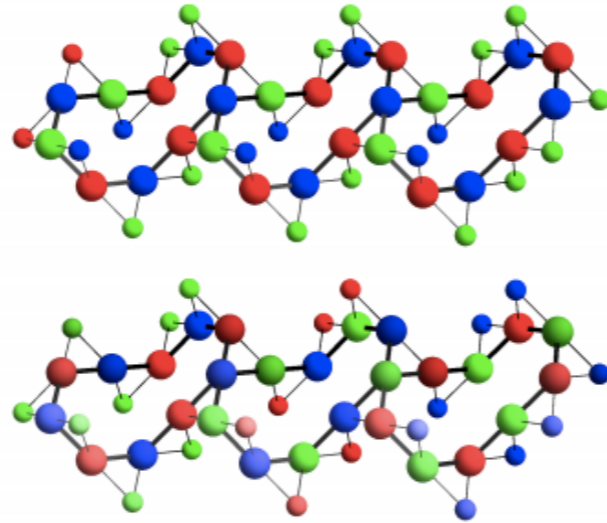
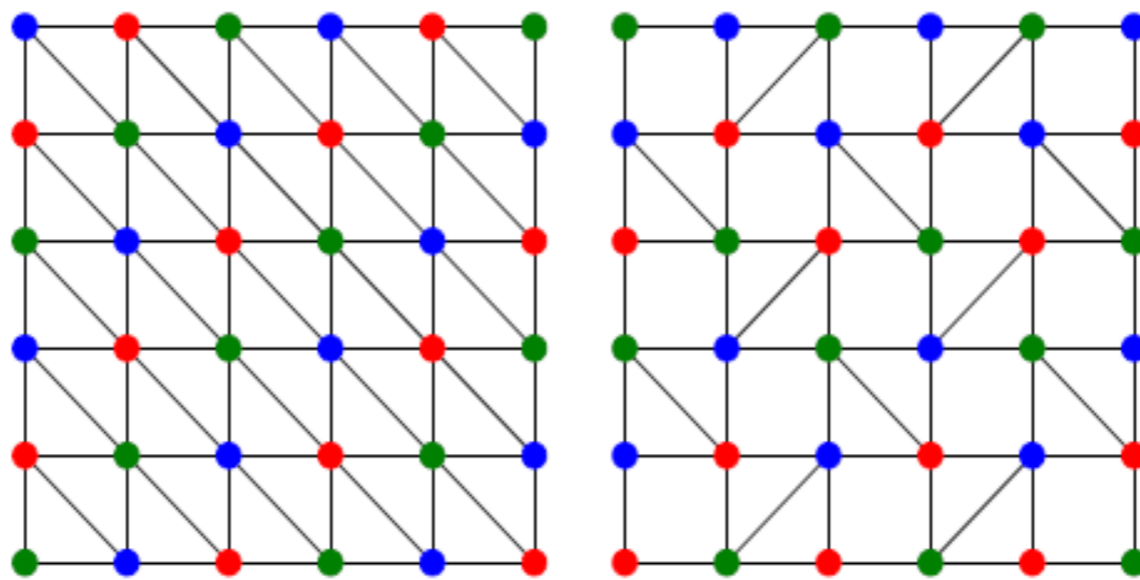
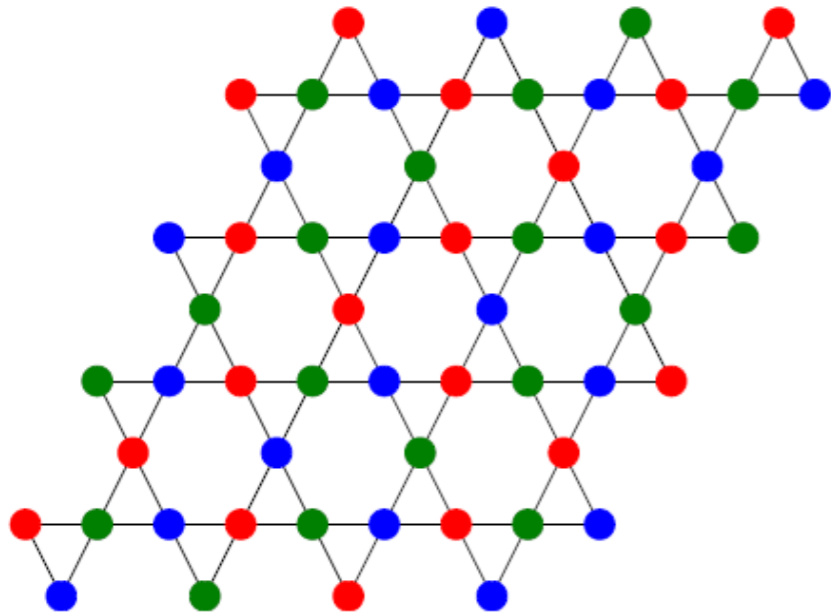


**Frustration Free!  
(but not commuting)**



$$H = \sum_{\Delta} H_{XXZ0}$$

$$|\psi\rangle \equiv \prod_s \otimes |C_s\rangle_s$$





Actually there are more ground states....

$$|\psi\rangle \equiv \prod_s \otimes |C_s\rangle_s \quad \text{This mixes } S_z \text{ sectors}$$



But the Hamiltonian doesn't.

$$|\psi^C\rangle \equiv P_{S_z} \left( \prod_{\text{valid}} \otimes |C_s\rangle \right) \quad \text{So projecting to } S_z \text{ sectors are ground states.}$$

Roughly, each color gives  $N$  ground states (one per  $S_z$  sector)

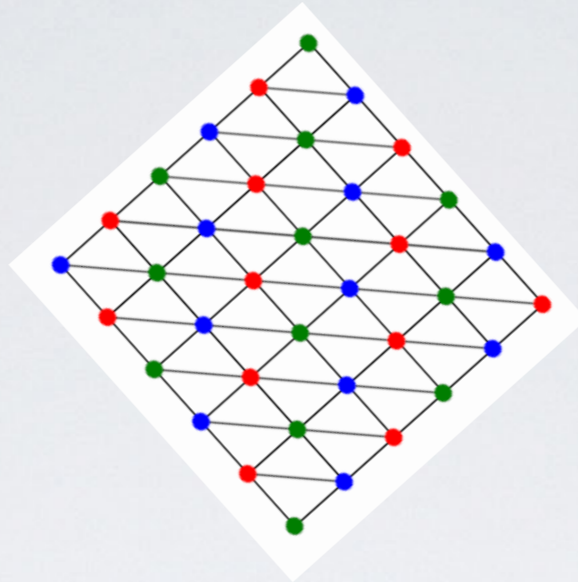
*(A bit of a lie because colors are non-orthogonal and may be more-so after projection)*

$$\left\{ P_{S_z=0} \left[ \begin{array}{c} \text{Lattice} \end{array} \right], P_{S_z=1} \left[ \begin{array}{c} \text{Lattice} \end{array} \right], \dots, P_{S_z=N} \left[ \begin{array}{c} \text{Lattice} \end{array} \right] \right\}$$

# Can we use this to understand the triangular lattice?

**How many colorings?**

Only one (or two) colorings.



$$P_{S_z=0} \left[ \begin{array}{c} \text{Lattice Diagram} \end{array} \right]$$



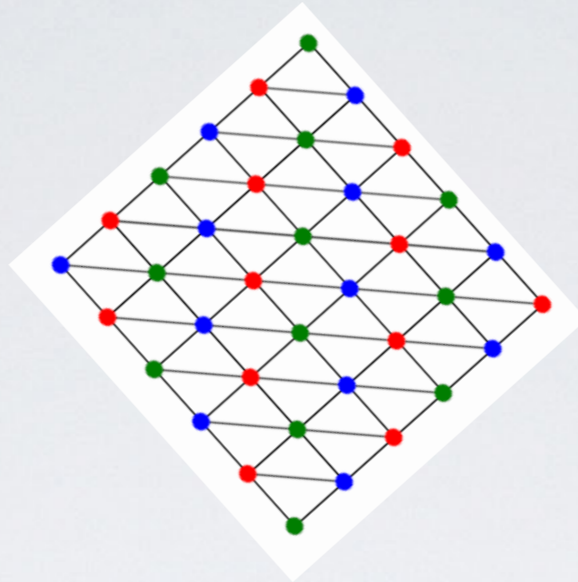
$$H = H_{\text{XXZ0}} + 1.5 \sum_{ij} S_i^z S_j^z$$



# Can we use this to understand the triangular lattice?

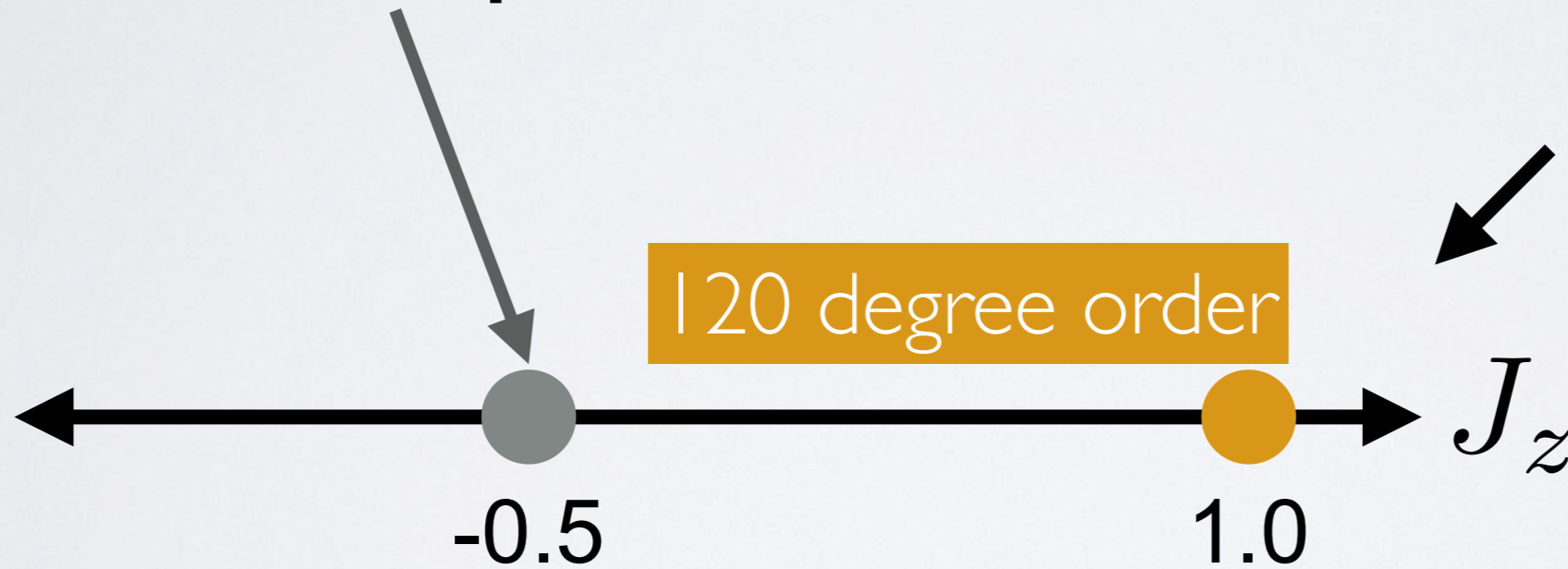
**How many colorings?**

Only one (or two) colorings.



**'Linear Degenerate'  
over polarization**

$$P_{S_z=0} \left[ \begin{array}{c} \text{Lattice Diagram} \end{array} \right]$$



-0.5  
**XXZ0**

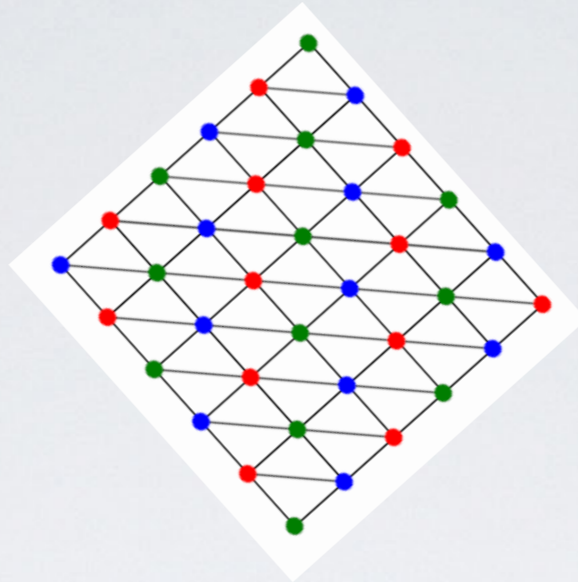
1.0  
**Heisenberg**

$$H = H_{\text{XXZ0}} + 1.5 \sum_{ij} S_i^z S_j^z$$

# Can we use this to understand the triangular lattice?

## How many colorings?

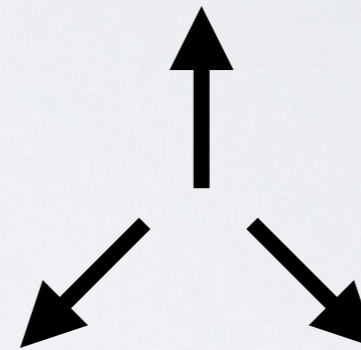
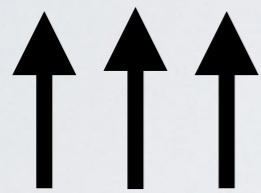
Only one (or two) colorings.



$$P_{S_z=N} \left[ \begin{array}{c} \text{Lattice Diagram} \end{array} \right]$$

'Linear Degenerate'  
over polarization

$$P_{S_z=0} \left[ \begin{array}{c} \text{Lattice Diagram} \end{array} \right]$$



Ferromagnet

120 degree order



-0.5

1.0

**XXZ0**

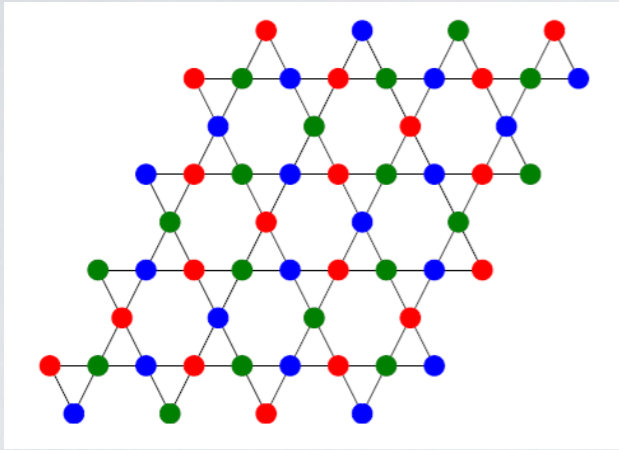
**Heisenberg**

$$H = H_{\text{XXZ0}} - 0.1 \sum_{ij} S_i^z S_j^z$$

$$H = H_{\text{XXZ0}} + 1.5 \sum_{ij} S_i^z S_j^z$$

# Can we use this to understand the kagome lattice?

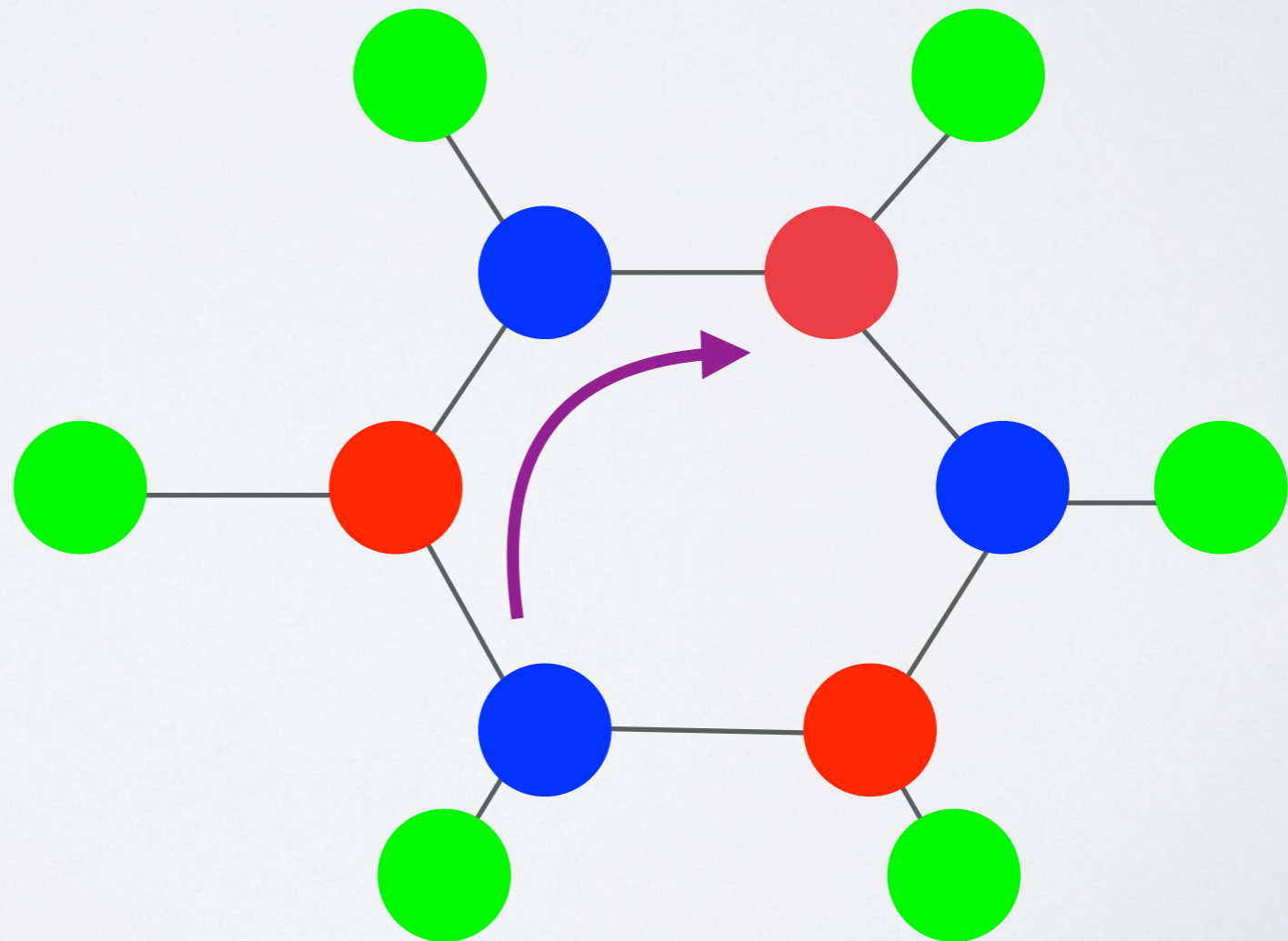
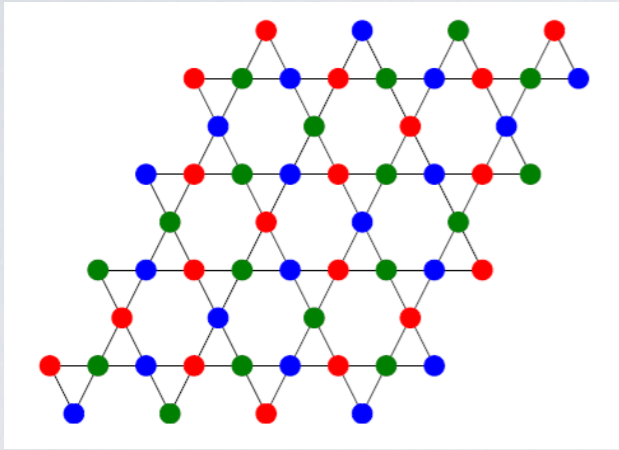
*How many colorings?*





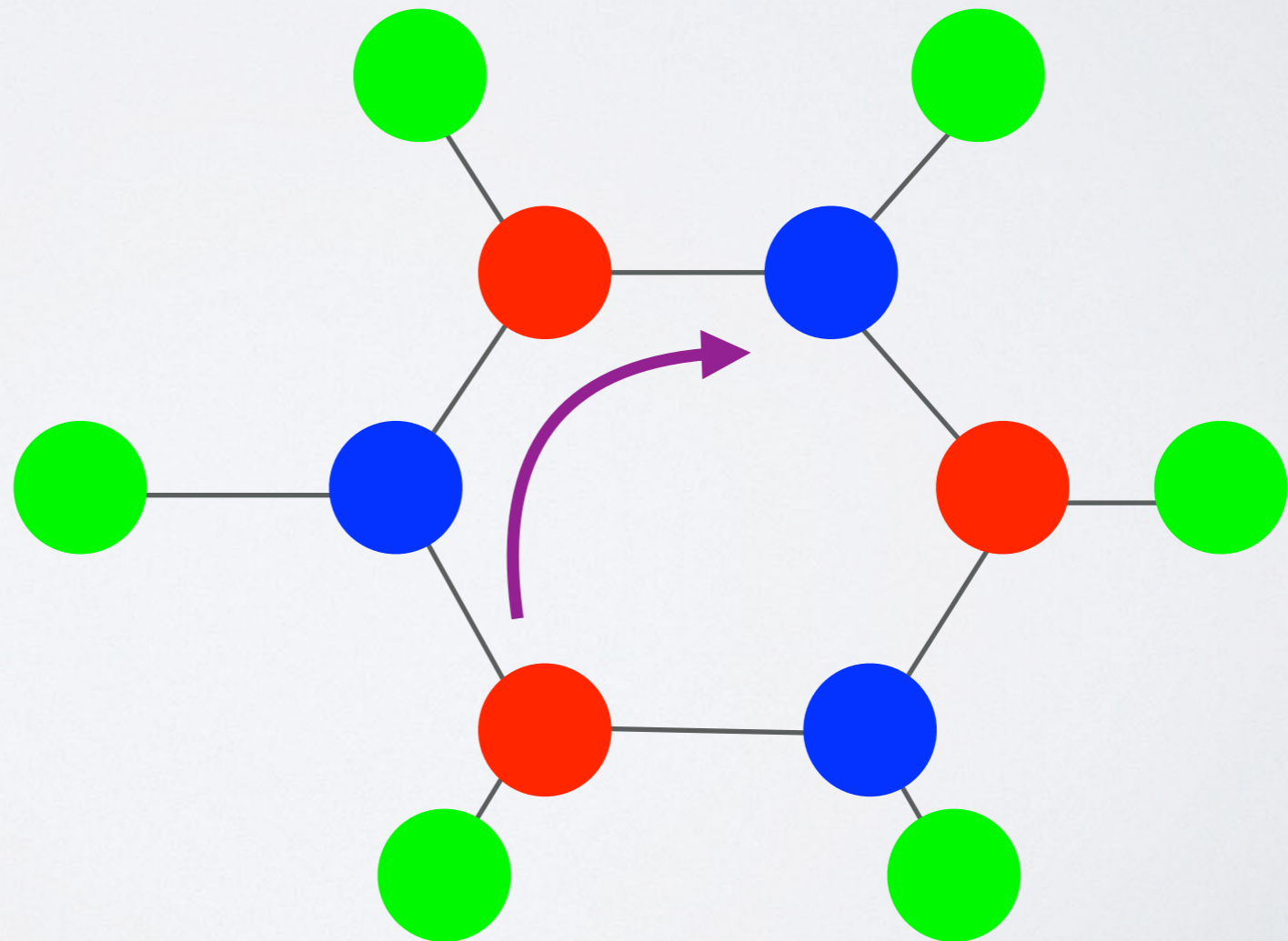
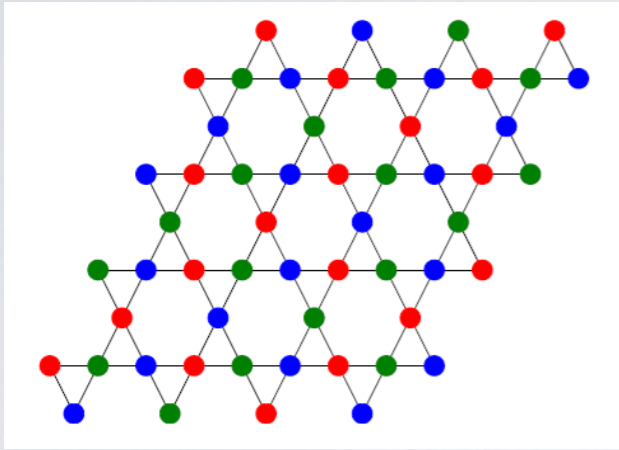
# Can we use this to understand the kagome lattice?

*How many colorings?*



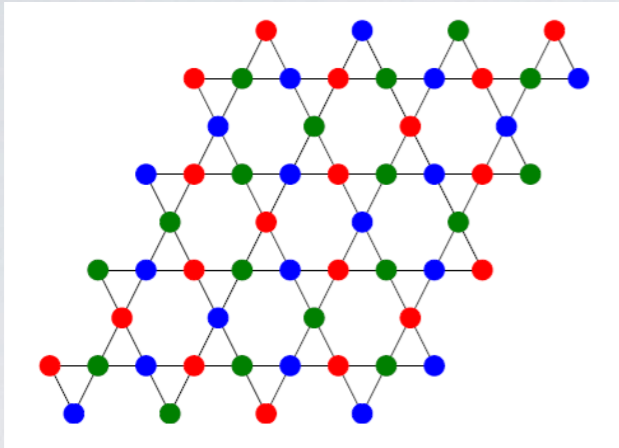
# Can we use this to understand the kagome lattice?

*How many colorings?*



# Can we use this to understand the kagome lattice?

*How many colorings?*



**An exponential number of colorings!**

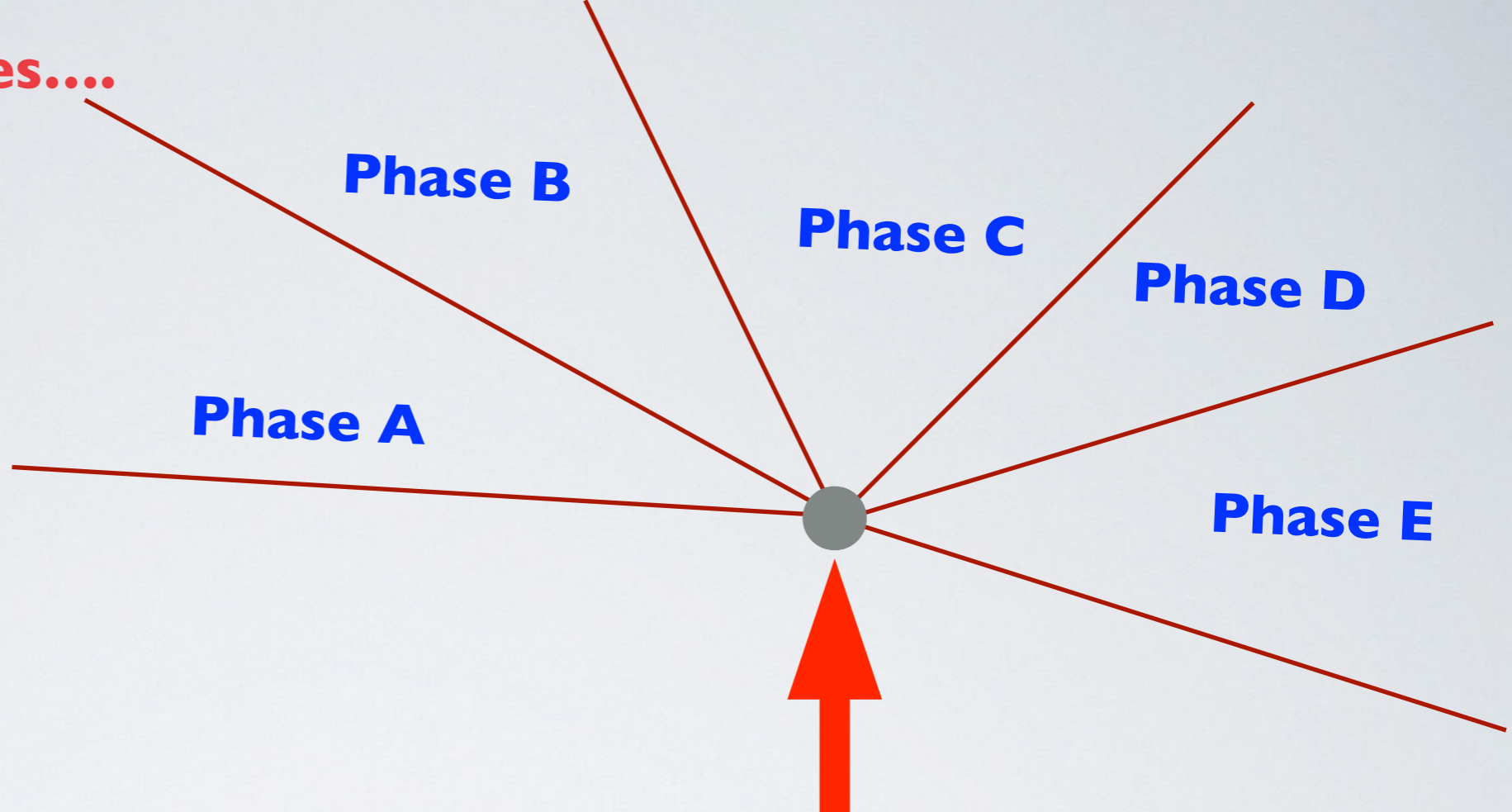
$$1.208^N \text{ (from Baxter)}$$

**But much fewer than Ising configurations....**

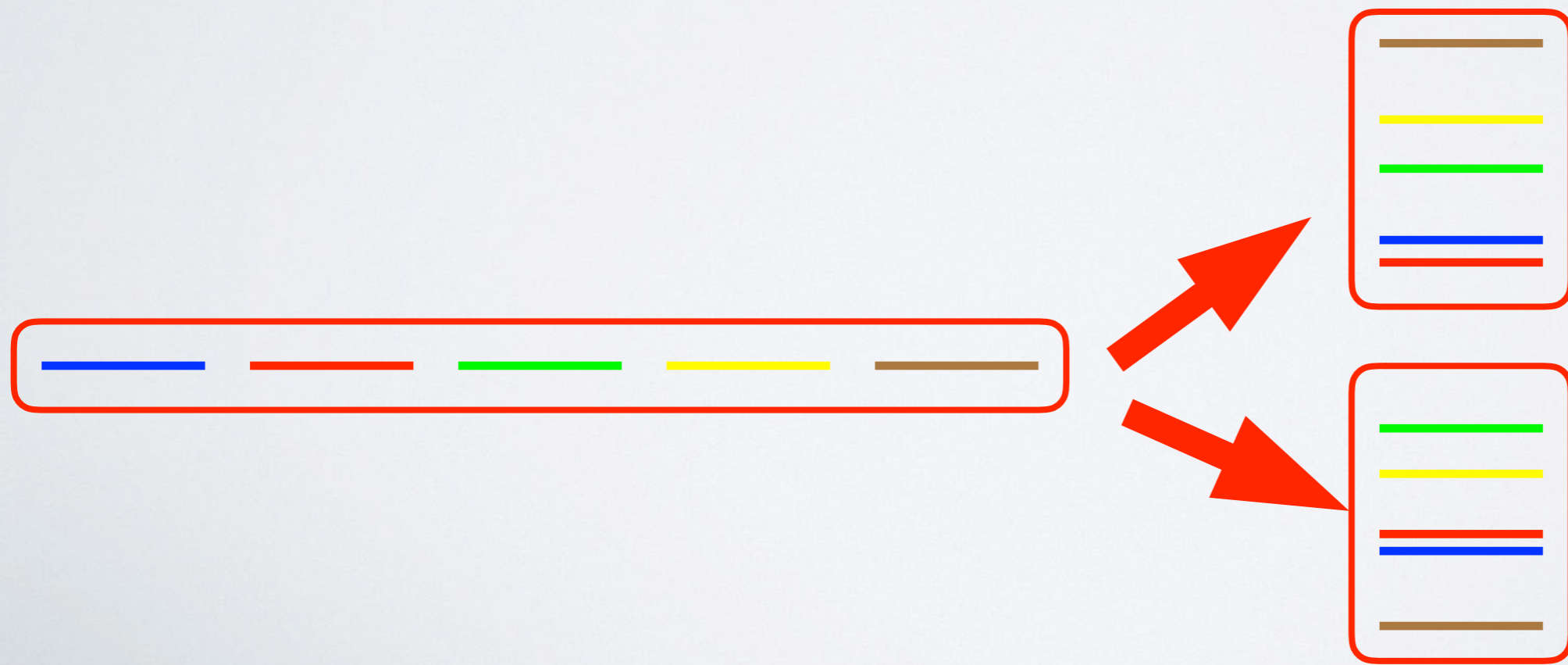
Lattice	Ising configs	Colorings
2x2x3	924	8
3x2x3	48620	16
4x2x3	2.7 million	32



**Connect to known phases....**

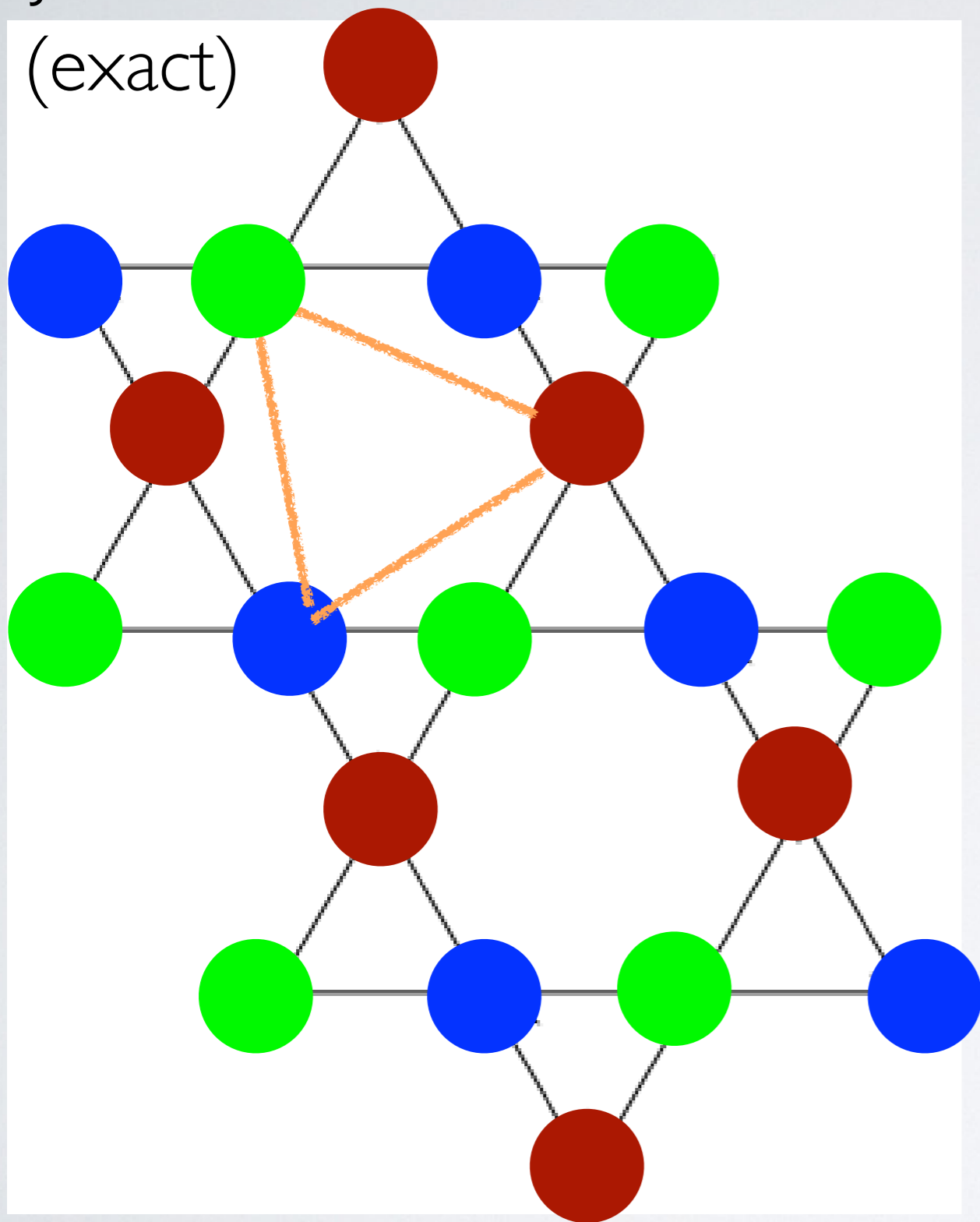


**The mother of all phases?**



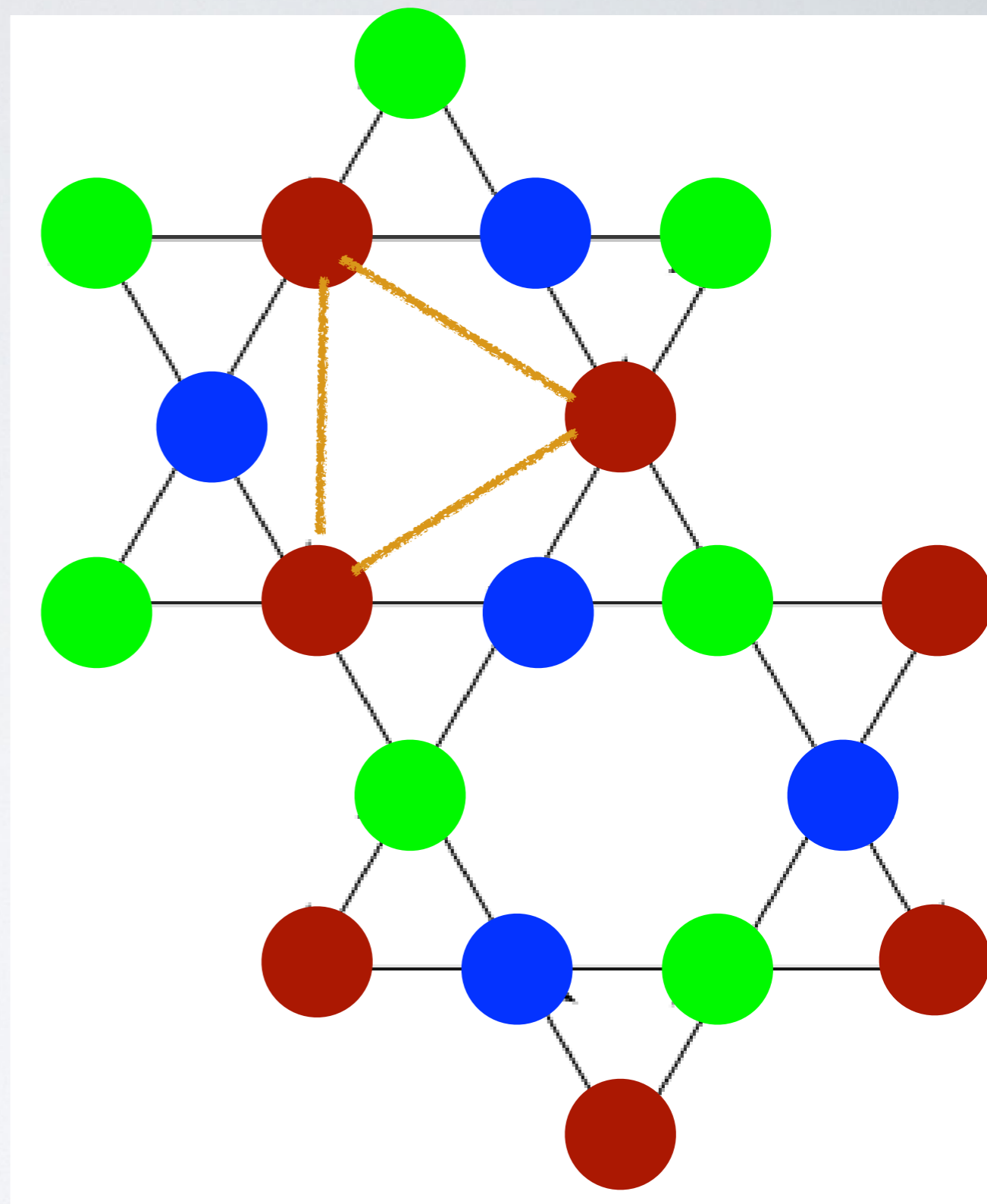
$J_2 > 0$

(exact)

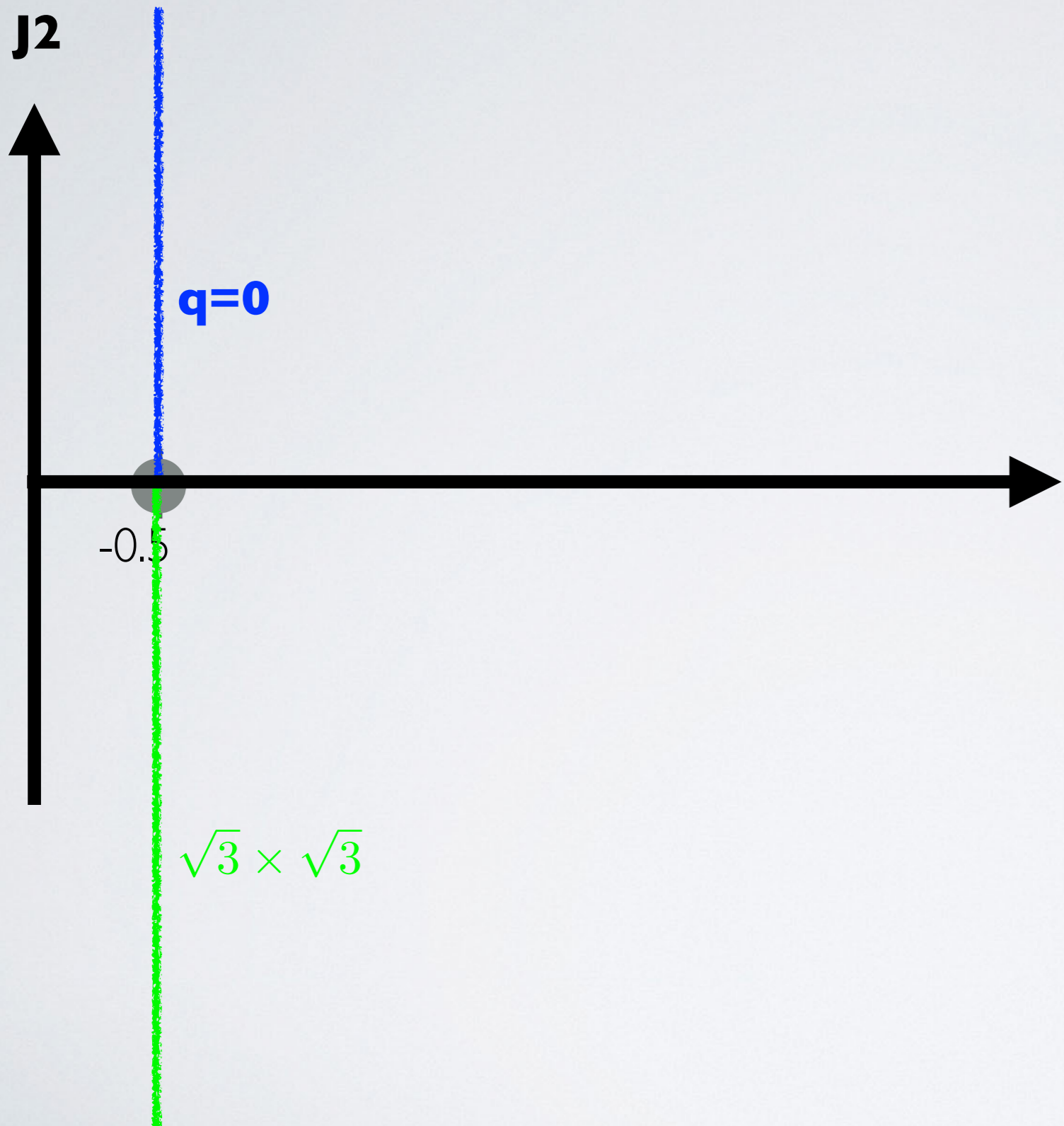


$q=0$

$J_2 < 0$

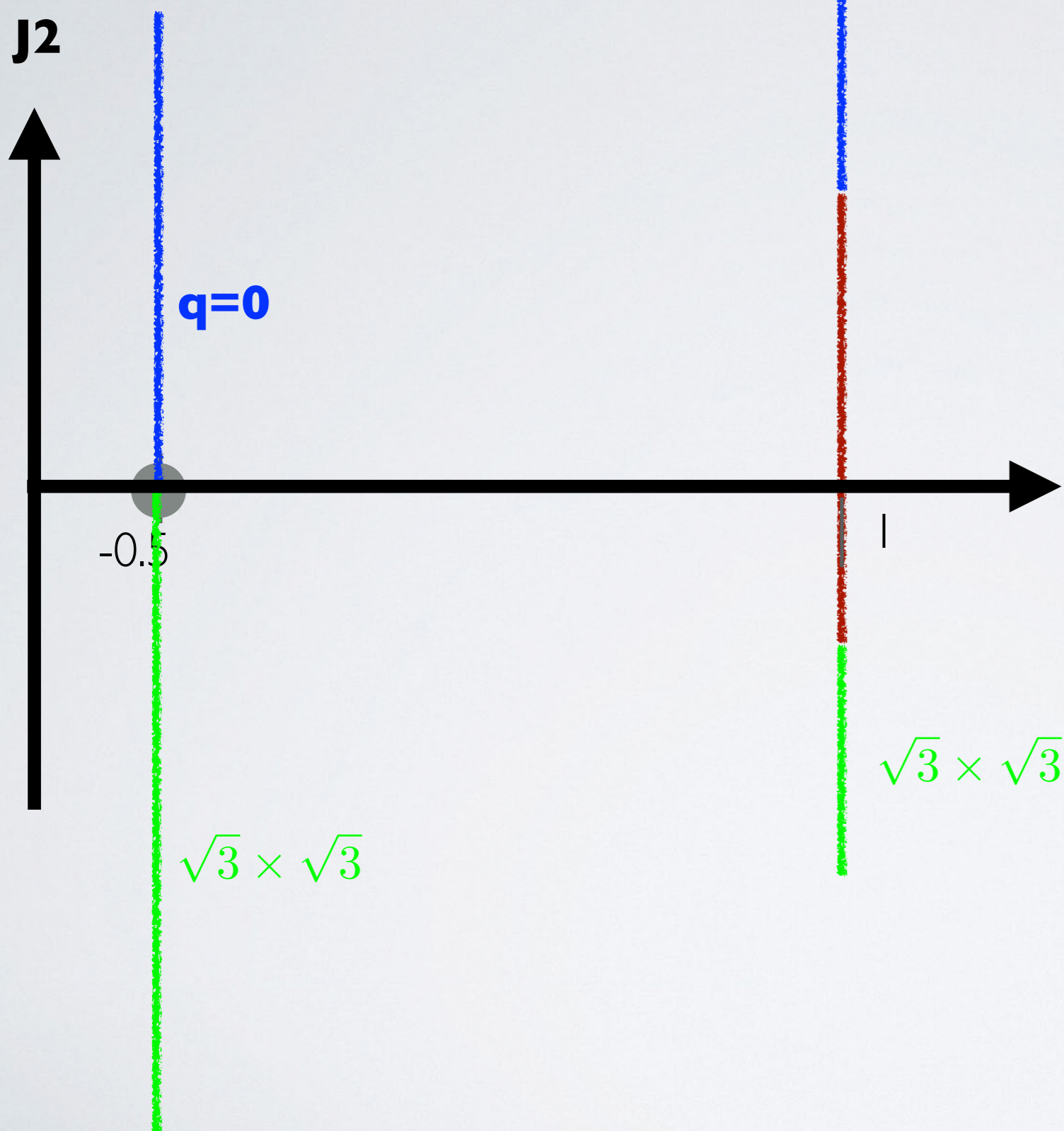


$\sqrt{3} \times \sqrt{3}$

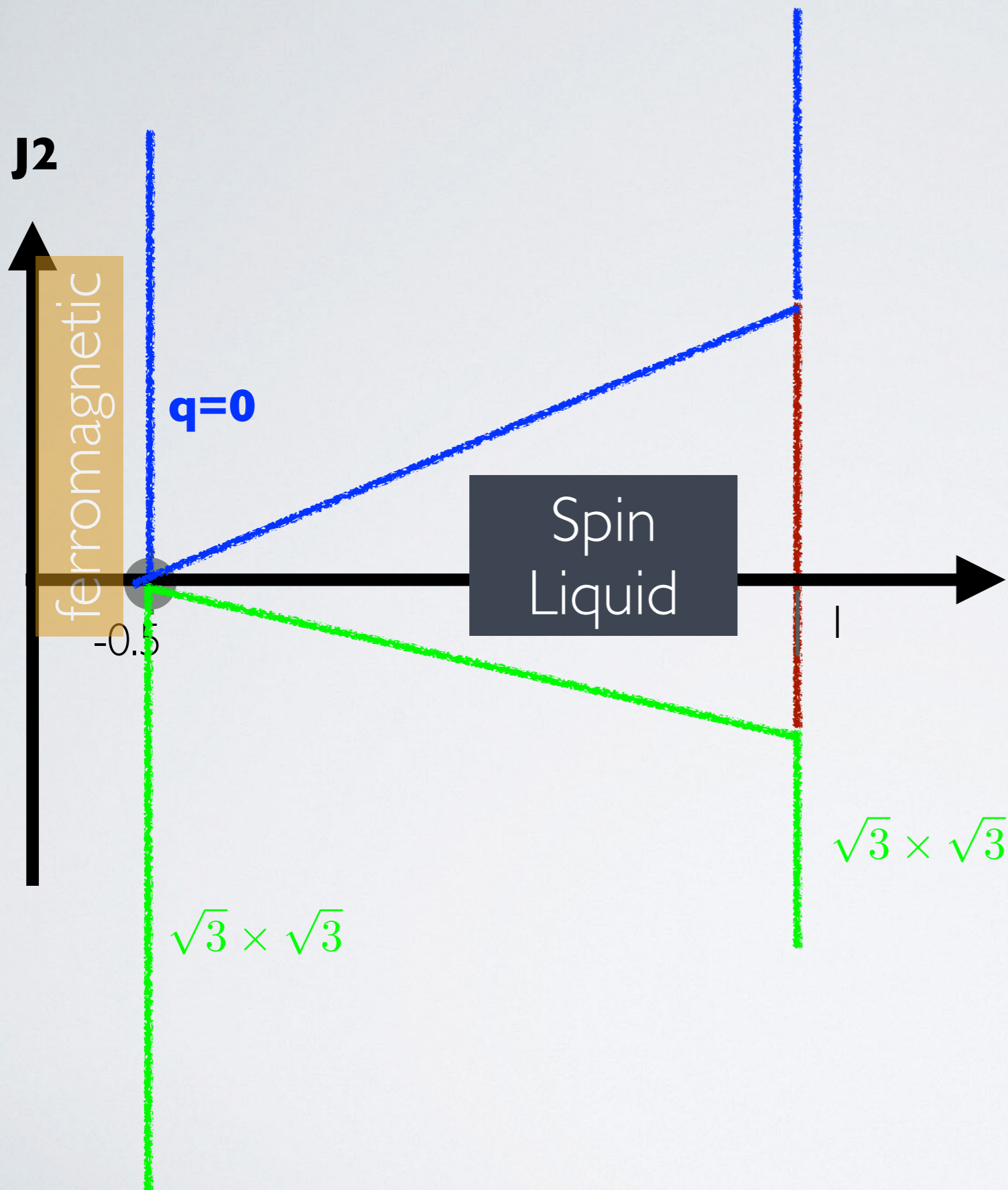




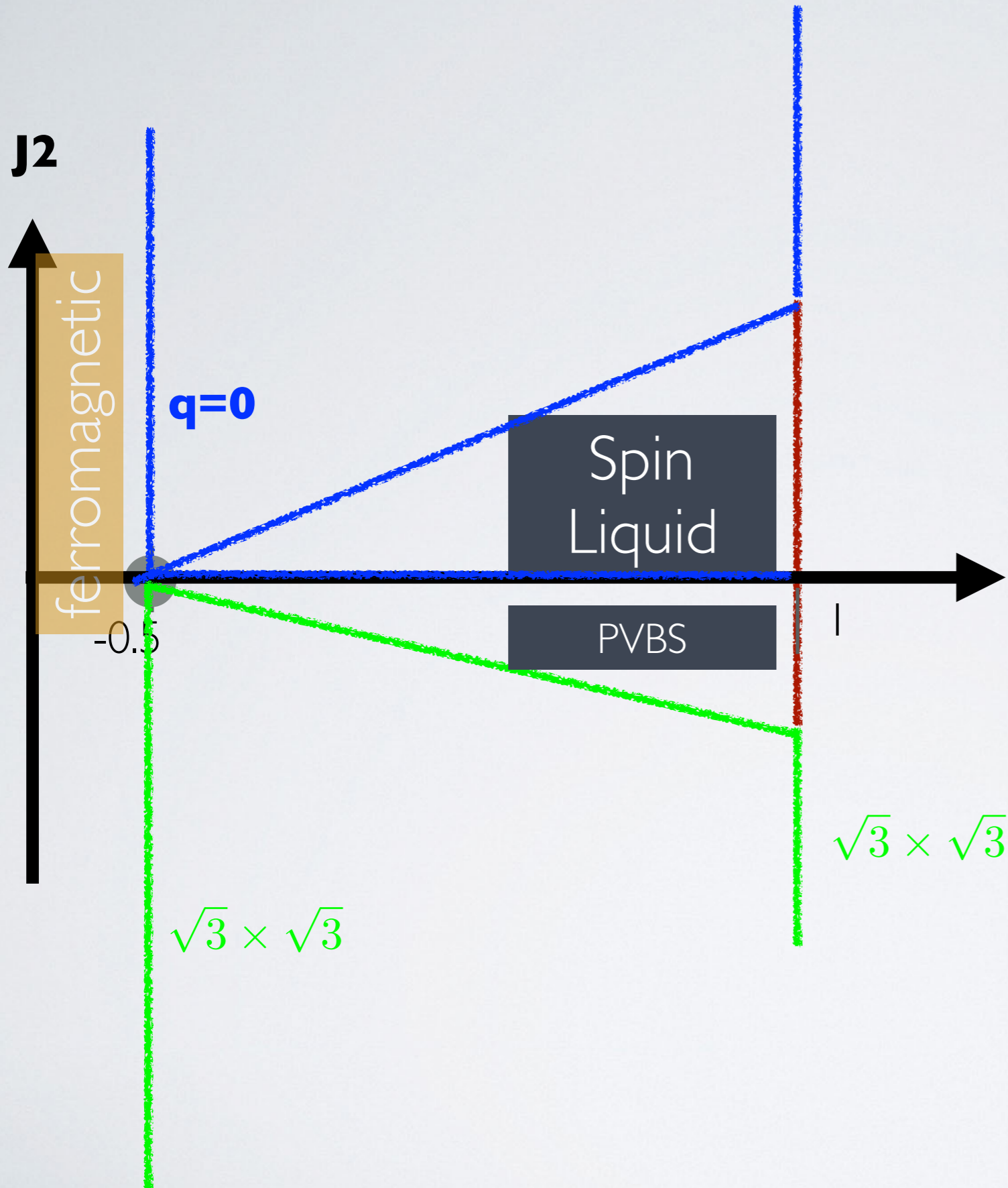
$$S_z = 0$$



$$S_z = 0$$



$$S_z = 0$$



ferromagnetic

$q=0$

Spin  
Liquid

PVBS

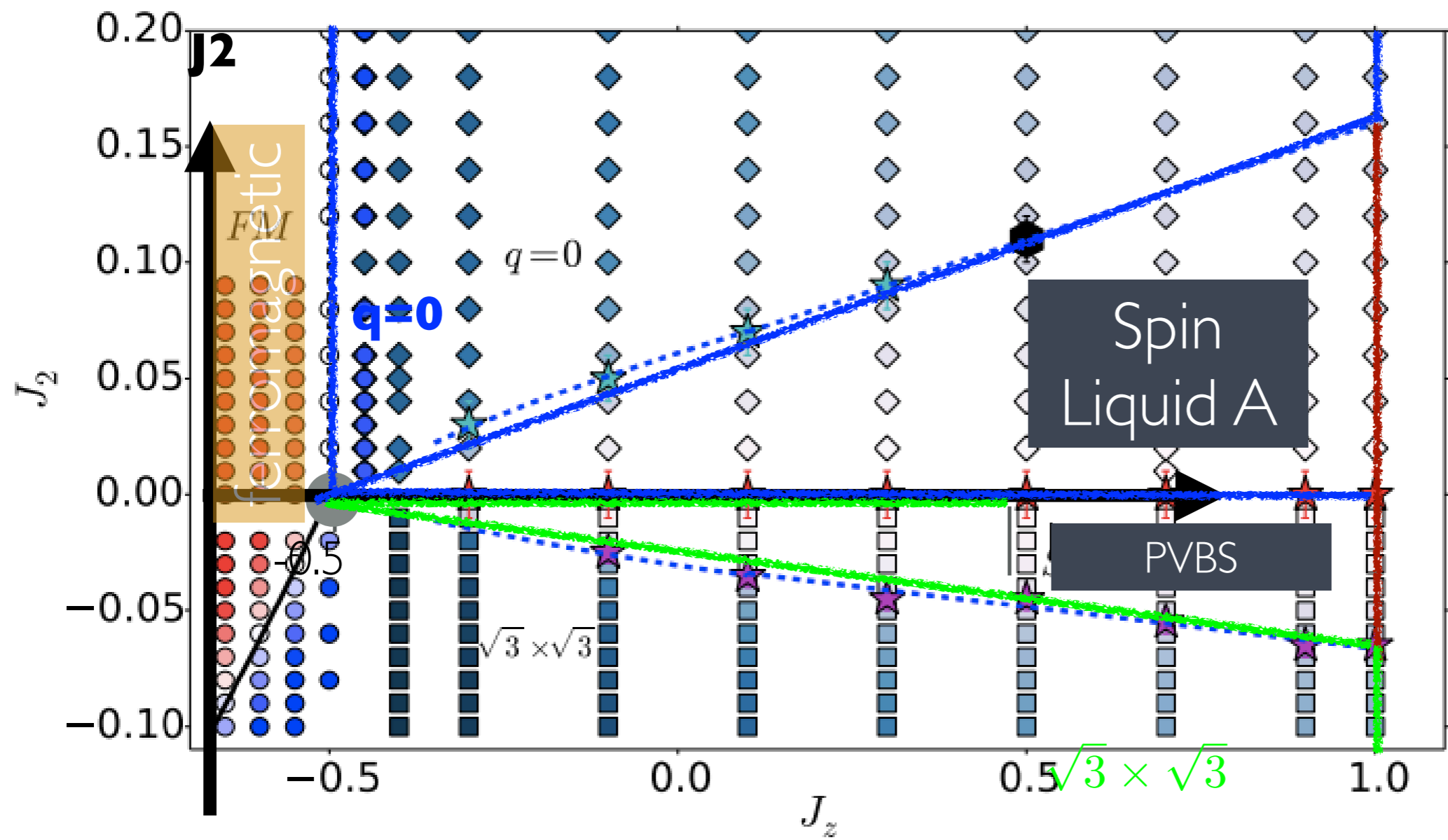
-0.5

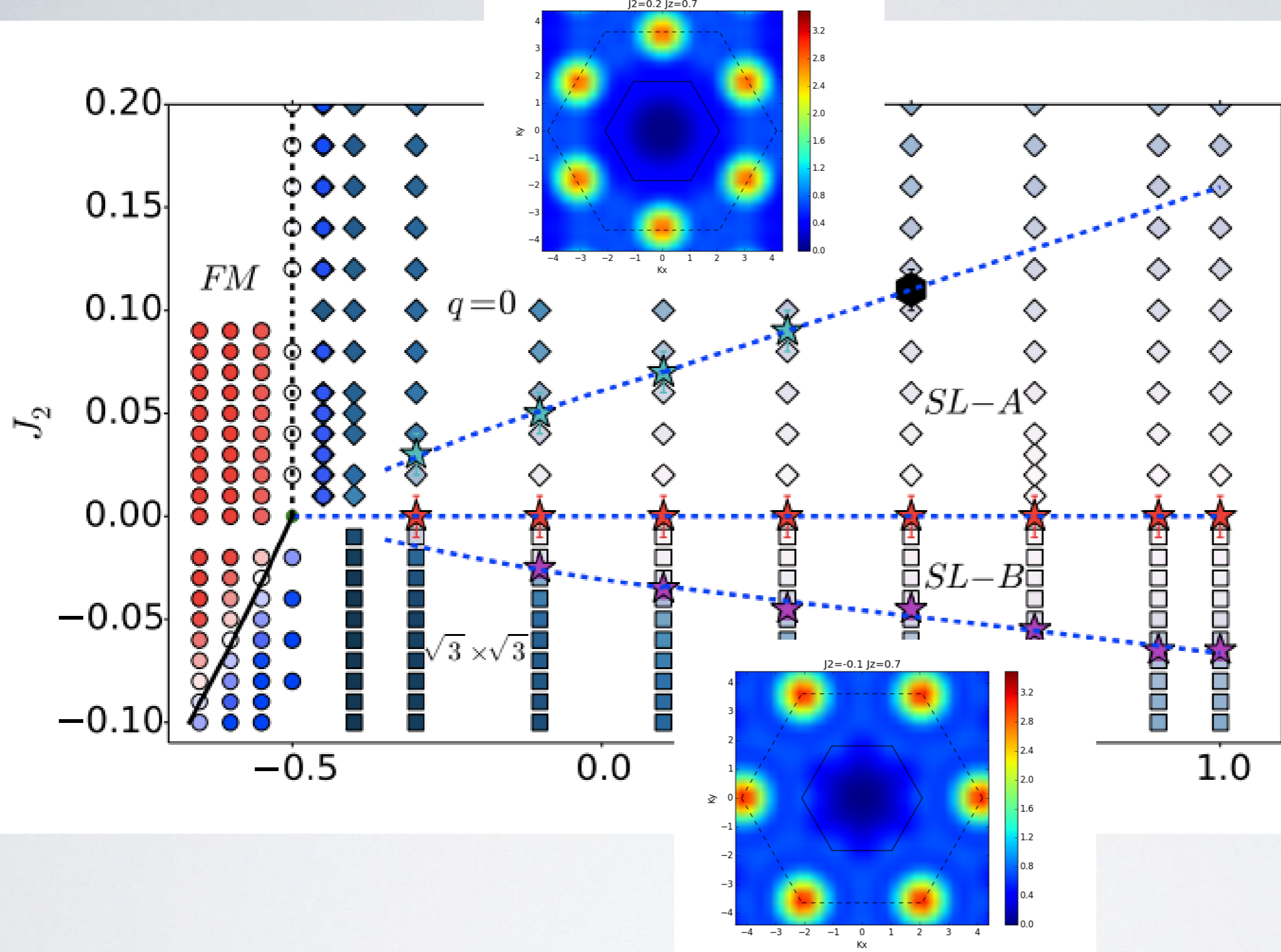
$\sqrt{3} \times \sqrt{3}$

$\sqrt{3} \times \sqrt{3}$

1







Q: Why co-planar states?

Colorings are all co-planar

Q: Why these co-planar states?

Fixed by colorings which satisfy  $J_1$ - $J_2$

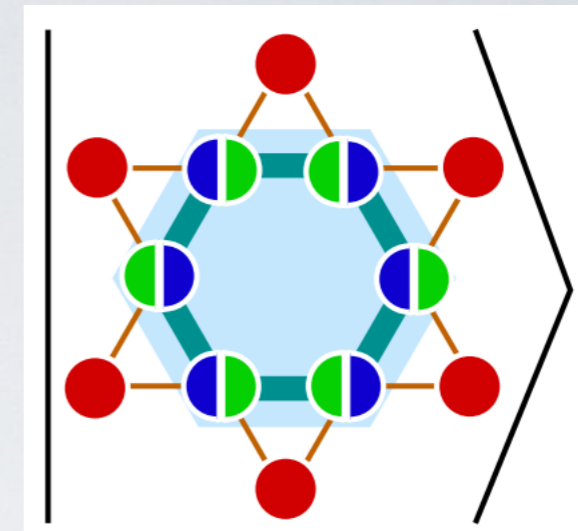
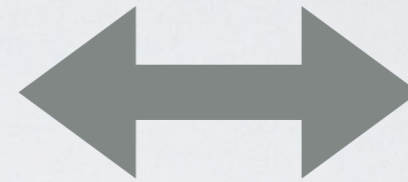
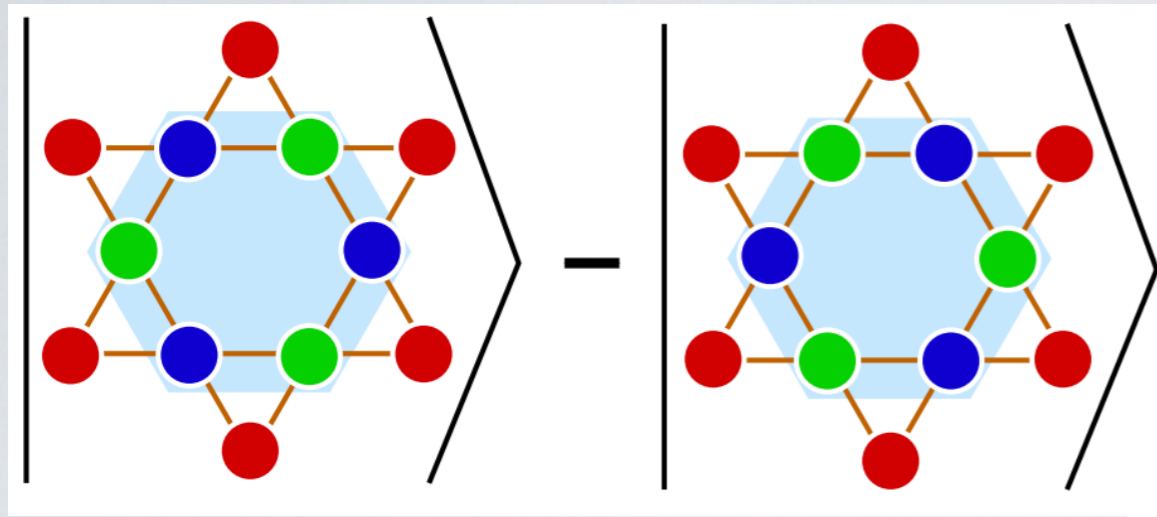
Q: Why spin-liquids?

Q: Why so many competing phases?

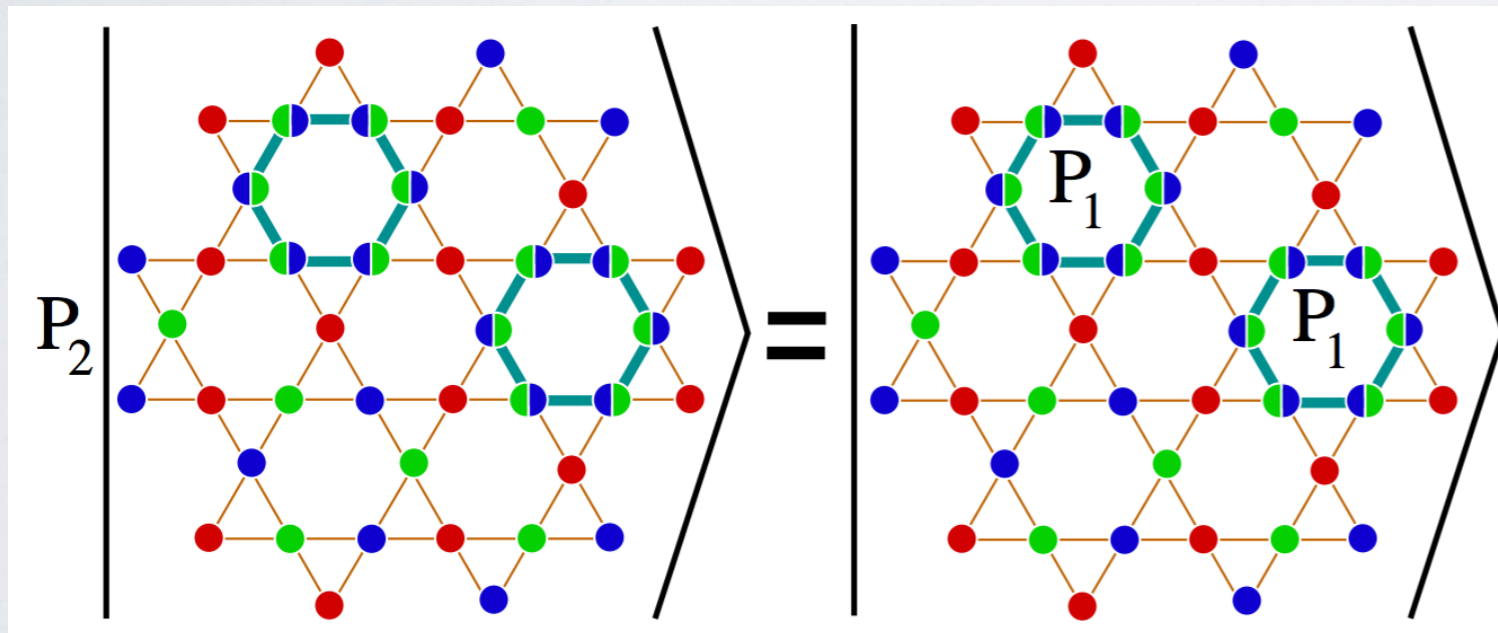
Q: Why low-energy mess?

Exponential Degeneracy

# Kagome at high magnetization sector.



Resonating Color Loops  
(like RVB)



$$\underline{H_{XXZ0}} + J_z \sum_{\langle i,j \rangle} \underline{\sigma_i^z \sigma_j^{z+1}}$$

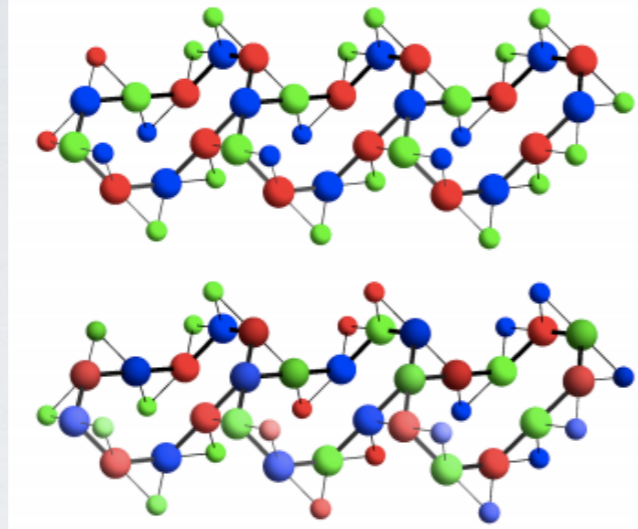
Frustration Free...

*Exact ground state is in the coloring space*



# Can we use this to understand the hyper-kagome lattice?

What's known: *experimental evidence of a spin-liquid.*

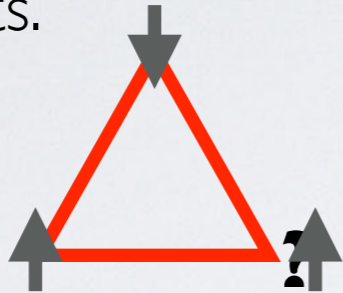


Also exponential degeneracy

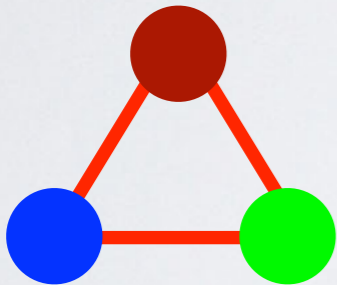
# Conclusions

XXZ0 controls the physics of the Heisenberg point on lattices of pasted triangles in the way that the Ising limit doesn't.

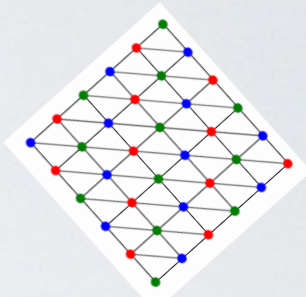
The story of frustration is not one of triangles which can't satisfy up-up-down constraints.



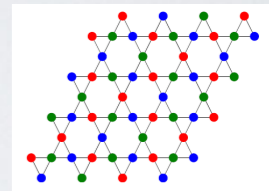
Instead, the story of frustrated magnetism is really one of coloring.



A single coloring which controls the triangular lattice.



And an exponential number of colorings which controls the kagome lattice.



From which all the known phases (and I conjecture arbitrarily many more) arise.