QUANTUM COLORING

University of Illinois at Urbana Champaign
with Hitesh Changlani, Krishna Kumar, Eduardo Fradkin

PCTS@10

Frustrated Magnetism

Spin 1/2 quantum Hamiltonian's

$$H_{xy} = \sum_{\langle i,j \rangle} S_i^x S_j^x + S_i^y S_j^y$$

$$H_{xxz} = H_{xy} + J_z \sum_{ij} S_i^z S_j^z$$

→ Ground State

An old subject.

On the ground state properties of the anisotropic triangular antiferromagnet

By P. Fazekas† and P. W. Anderson‡
Cavendish Laboratory, Cambridge, England

[Received 24 May 1974]

Simple Variational Wave Functions for Two-Dimensional Heisenberg Spin-$\frac{1}{2}$ Antiferromagnets

David A. Huse and Veit Elser
AT&T Bell Laboratories, Murray Hill, New Jersey 07974

(Received 24 February 1988)

PHYSICAL REVIEW B, VOLUME 63, 224401

Ising models of quantum frustration

R. Moessner and S. L. Sondhi
artment of Physics, Princeton University, Princeton, New J

(Received 14 November 2000; published 4 May 2001

A correlator product state study of molecular magnetism in the giant Keplerate $Mo_{72}Fe_{30}$

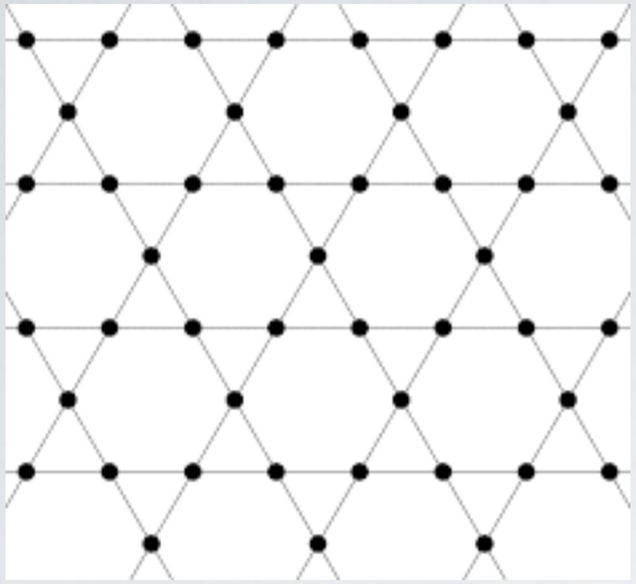
Eric Neuscamman* and Garnet Kin-Lic Chan†

I'm going to tell you about an interesting analytically solvable point (on a special Hamiltonian in ***triangular, Shastry-Sutherland, kagome, etc.)*** that could (should?) have been been discovered 20 years ago.

and tell you that this interesting point uncovers supports extensive degeneracy in kagome
(a many body flat band)

which might help us understand exotica in kagome.

After many years, we know that certain spin systems support spin liquid phases….



Z2 spin liquid heisenberg (White/Huse); xy (..)

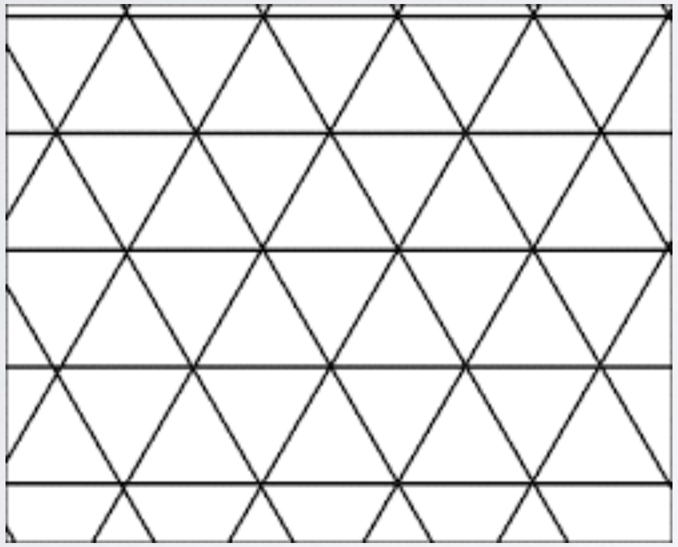
Chiral spin liquid: 2/3 plateau (this work)

1/3 plateau (Donna Sheng)

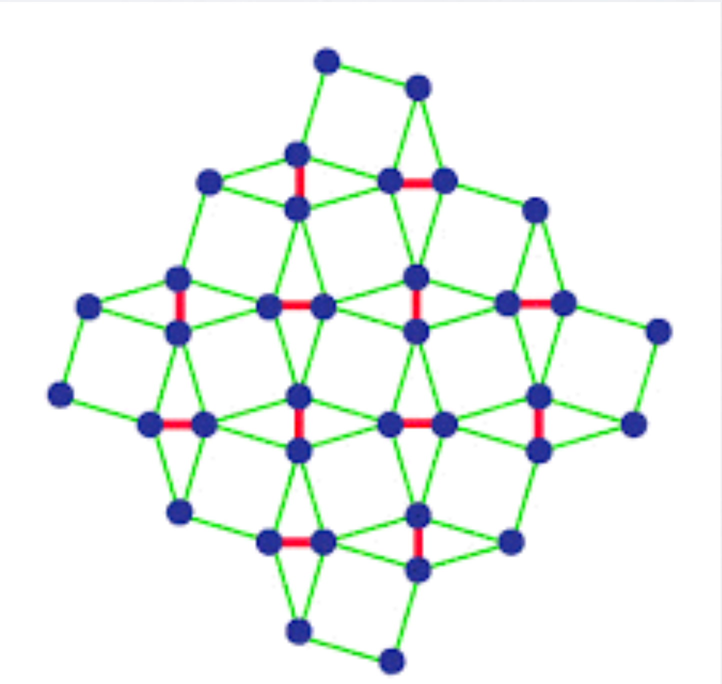
Sz=0 chiral (Bela Bauer, Andreas Ludwig)

Sz=0 J1,J2,J3 (Donna Sheng)

and that some systems do not support spin liquid phases…



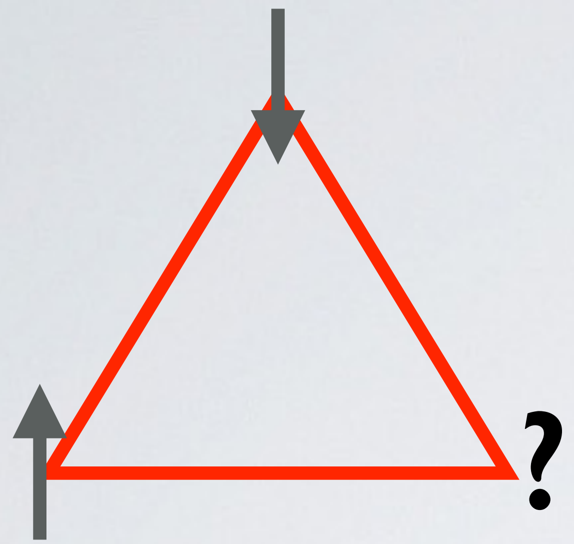
triangular
120 degree order



shastry-sutherland $J_2 = 2J_1$
dimer solid

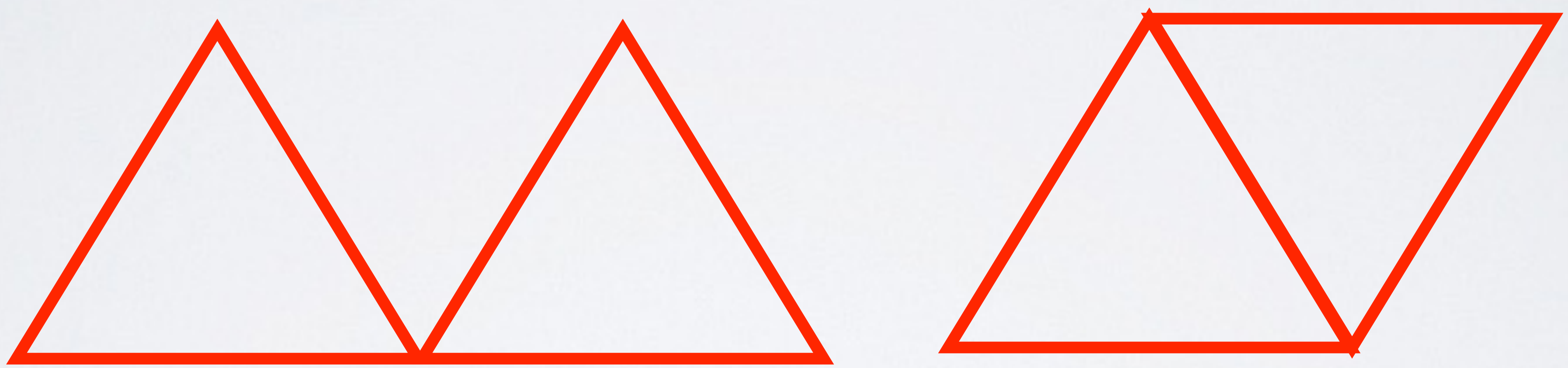
How should we understand this?

A common story is that from triangles come frustration….



$$H = J_z \sum_{ij} S_i^z S_j^z$$

When you paste together many triangles, there are many degenerate states



And this degeneracy gets resolved into a spin liquid.

But that doesn't really tell us why triangles are boring and kagome is interesting.

We could wave our hands and talk about amount of frustration but would like something more compelling.

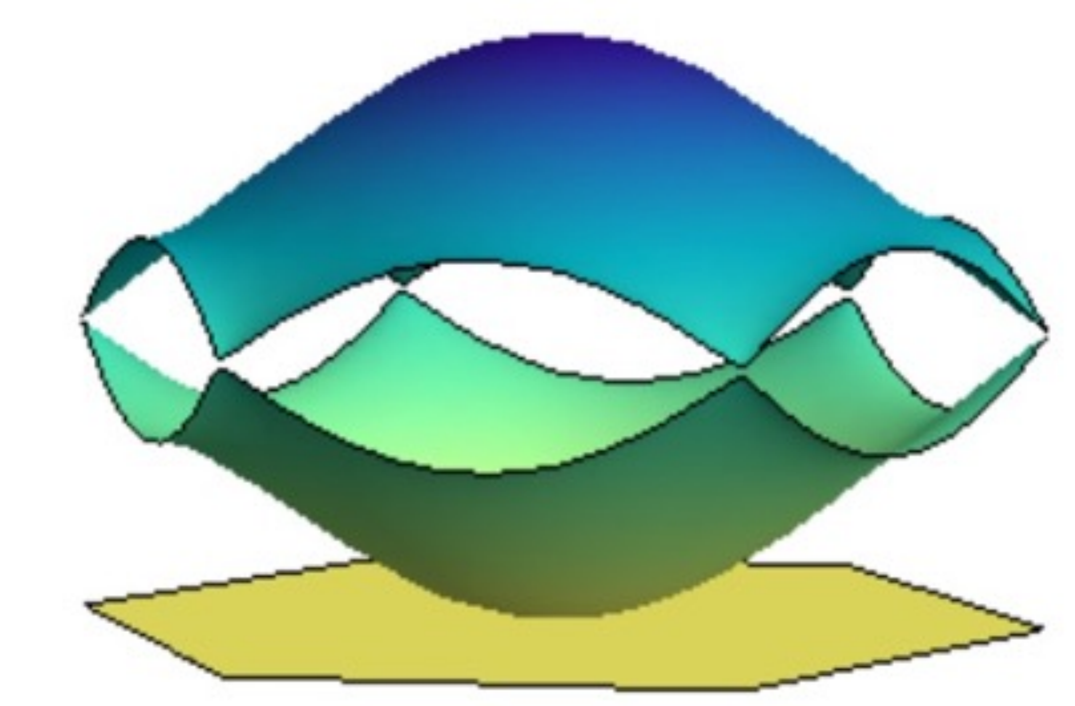
One difference, is that kagome has flat bands for one particle. Maybe this is a clue?

PHYSICAL REVIEW B **78**, 125104 (2008)

Band touching from real-space topology in frustrated hopping models

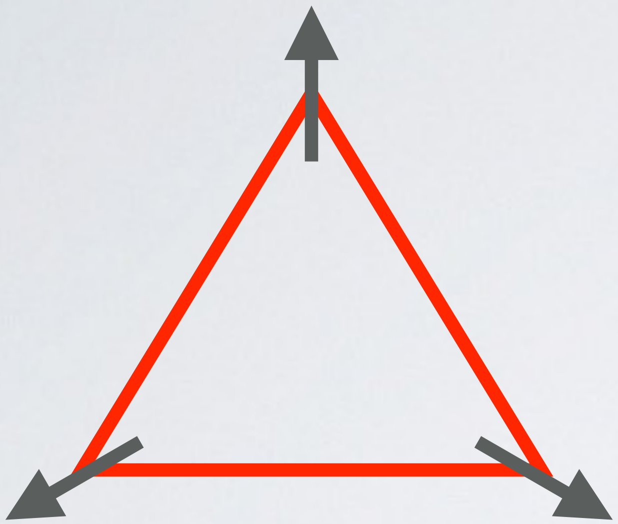
Doron L. Bergman,[1] Congjun Wu,[2] and Leon Balents[3]

$$\mathcal{H}_t = -t\sum_{\langle ij\rangle} (c_i^\dagger c_j + \text{H.c.}),$$



We've talked about frustration. Let's talk about the opposite.

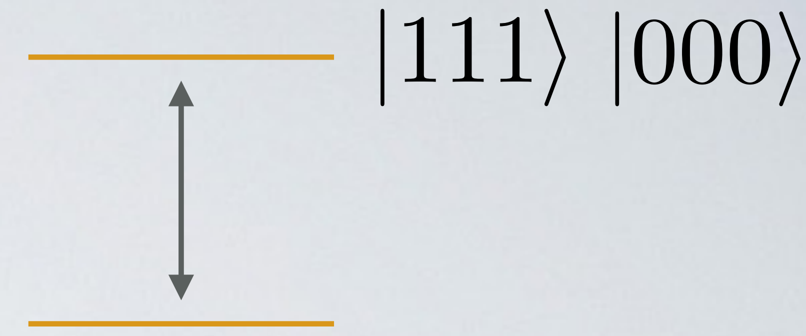
For a single triangle at the XY point, we can ***relieve frustration***.

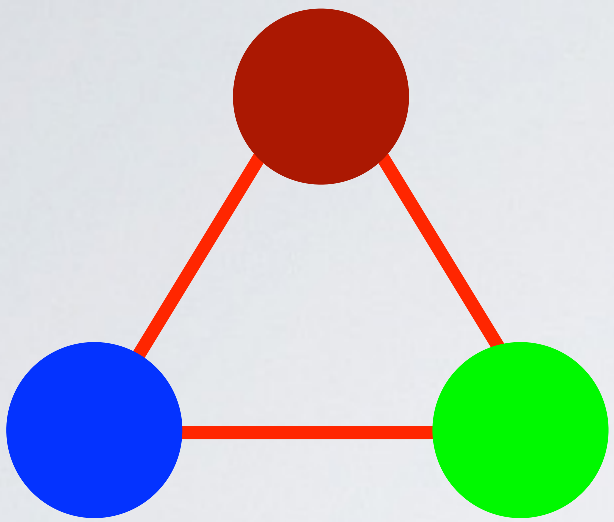


$$H = J_{xy} \sum_{ij} S_i^x S_j^x + S_i^y S_j^y$$

This is the exact ground state for (Sz=1/2) and everyone is happy

Relieving Frustration





$$H = J_{xy} \sum_{ij} S_i^x S_j^x + S_i^y S_j^y$$

ground state for $S_z = 1/2$

$$(|0\rangle + |1\rangle) \otimes (|0\rangle + \omega|1\rangle) \otimes (|0\rangle + \omega^2|1\rangle)$$

~~$|000\rangle + |111\rangle$~~ $+ |100\rangle + \omega|010\rangle + \omega^2|001\rangle + \ldots$

By projection

This is a high-energy eigenstate but projection removed it for us.

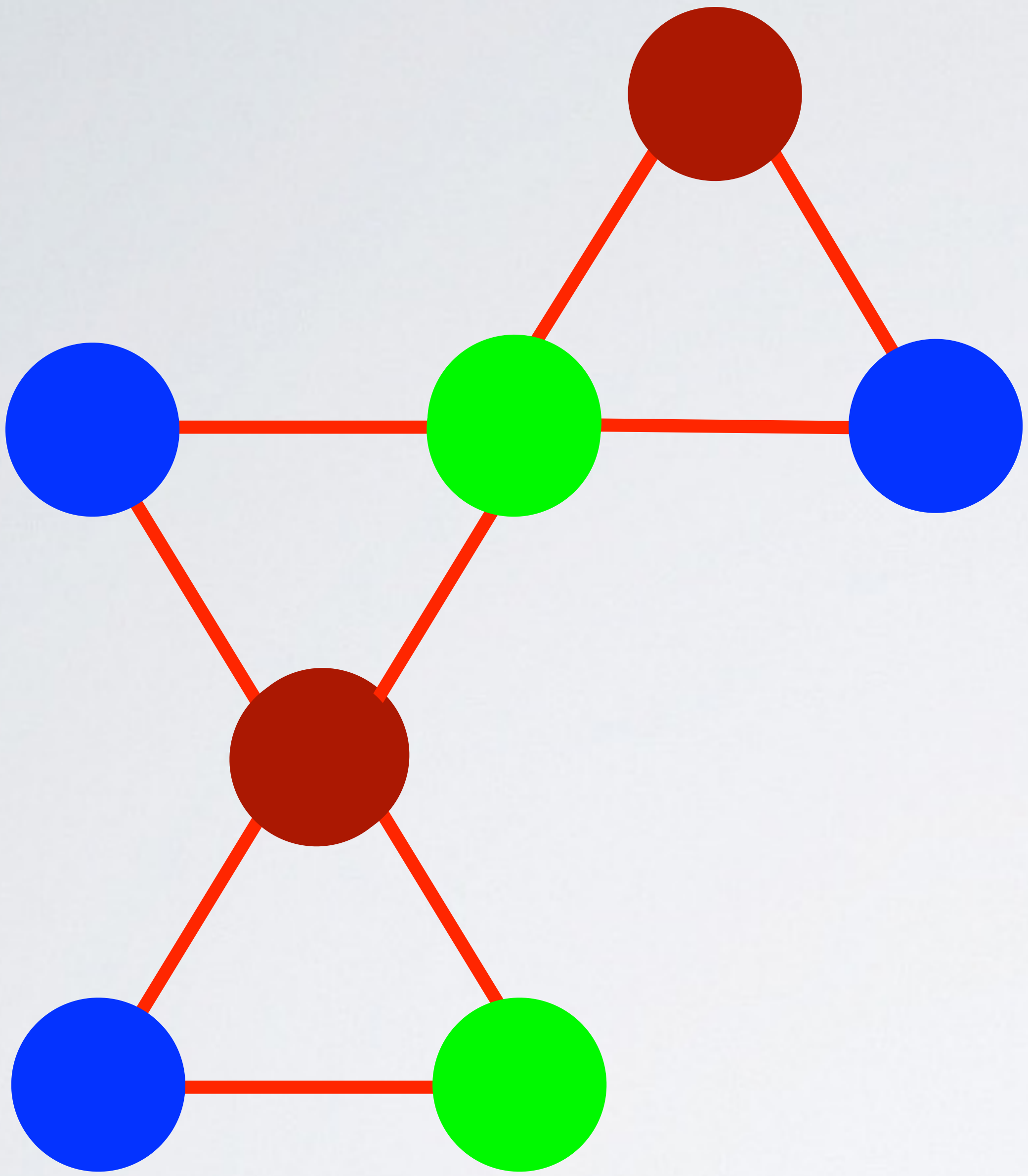
Define 3 "colors"

$$|a\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle + |\downarrow\rangle\Big)$$

$$|b\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle + \omega|\downarrow\rangle\Big)$$

$$|c\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle + \omega^2|\downarrow\rangle\Big)$$

Not Relieving Frustration



$$H = J_{xy} \sum_{ij} S_i^x S_j^x + S_i^y S_j^y$$

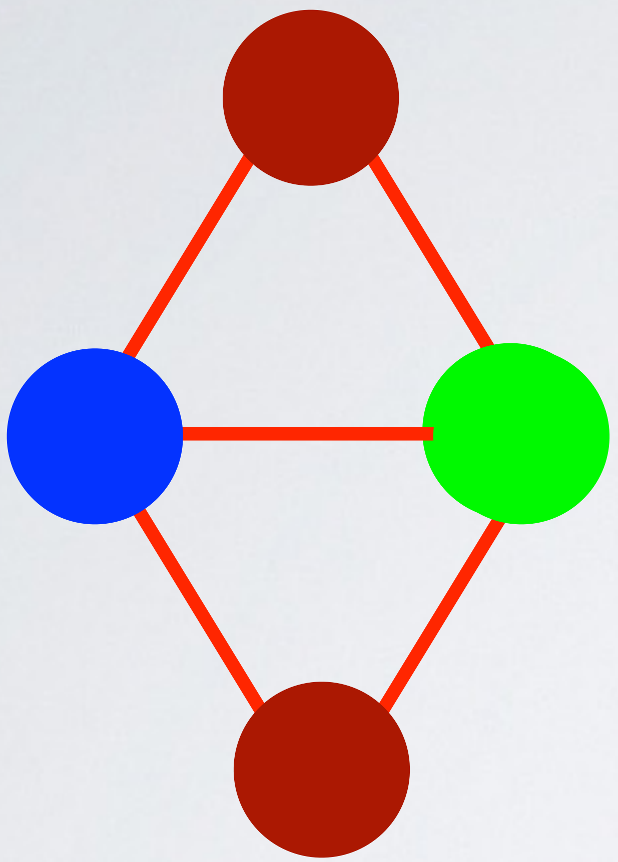
not ground state (for most Sz)

<u>*Why not?*</u>

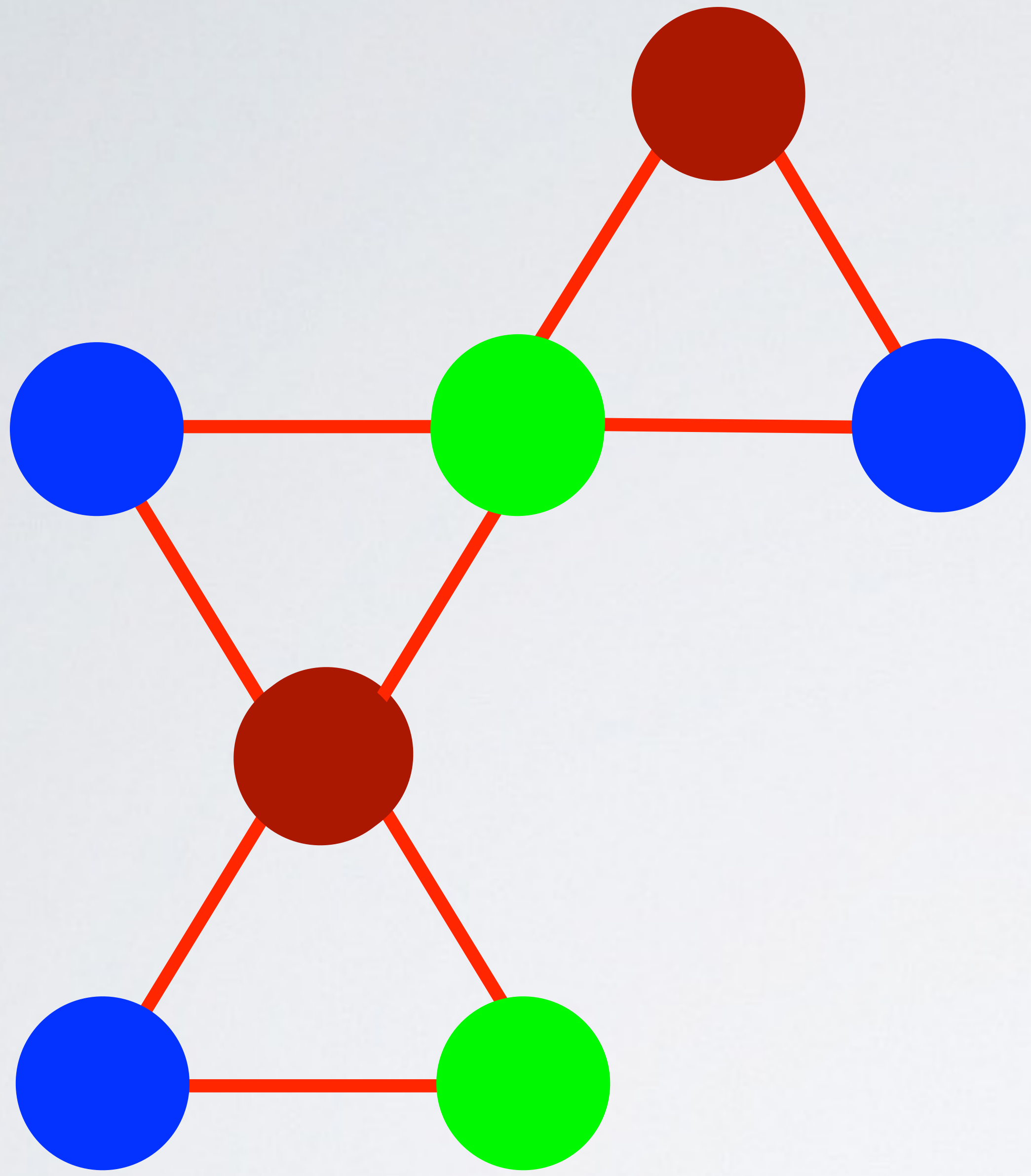
Simple answer 1: This product state gives some triangle all spin downs. That costs energy!

The projection doesn't remove 000 on all triangles

A side note…



This is morally the ground state for the XY model on the triangular lattice but it's not obvious why. Our answer must be too simple.



$$H = J_{xy} \sum_{ij} S_i^x S_j^x + S_i^y S_j^y$$

not ground state (for any Sz we've looked at)

Why not?

Simple answer 1: This product state gives some triangle all spin downs. That costs energy!

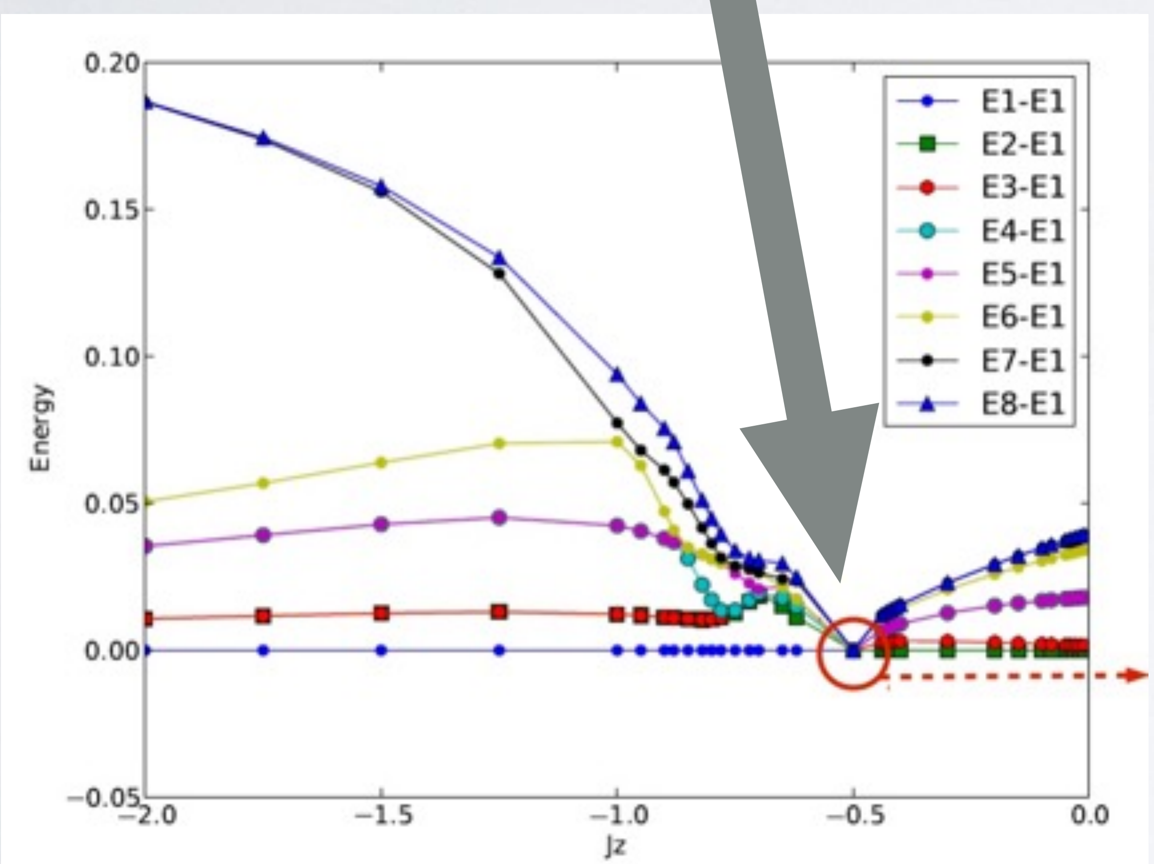
The projection doesn't remove 000 on all triangles

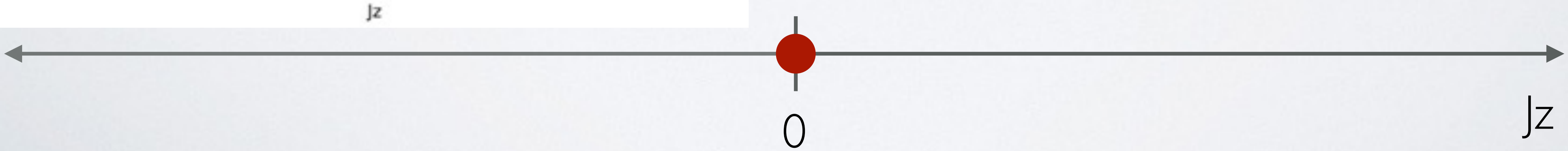
Simple answer 2: Right state, wrong Hamiltonian

$$H = \sum_{ij} S_i^x S_j^x + S_i^y S_j^y - 0.5 \sum_{ij} S_i^z S_j^z$$

massive exact degeneracy!

exactly -J/4





Who ordered that?

$$H = \sum_{ij} S_i^x S_j^x + S_i^y S_j^y - 0.5 \sum_{ij} S_i^z S_j^z$$

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$$H = \sum_{ij} S_i^x S_j^x + S_i^y S_j^y - 0.5 \sum_{ij} S_i^z S_j^z$$

$$|1\rangle \equiv |\uparrow\uparrow\uparrow\rangle$$

$$|2\rangle \equiv \frac{1}{\sqrt{3}}\Big(|\uparrow\uparrow\downarrow\rangle + \omega|\uparrow\downarrow\uparrow\rangle + \omega^2|\downarrow\uparrow\uparrow\rangle\Big)$$

$$|3\rangle \equiv \frac{1}{\sqrt{3}}\Big(|\uparrow\uparrow\downarrow\rangle + \omega^2|\uparrow\downarrow\uparrow\rangle + \omega|\downarrow\uparrow\uparrow\rangle\Big)$$

$$|4\rangle \equiv \frac{1}{\sqrt{3}}\Big(|\downarrow\downarrow\uparrow\rangle + \omega|\downarrow\uparrow\downarrow\rangle + \omega^2|\uparrow\downarrow\downarrow\rangle\Big)$$

$$|5\rangle \equiv \frac{1}{\sqrt{3}}\Big(|\downarrow\downarrow\uparrow\rangle + \omega^2|\downarrow\uparrow\downarrow\rangle + \omega|\uparrow\downarrow\downarrow\rangle\Big)$$

$$|6\rangle \equiv |\downarrow\downarrow\downarrow\rangle$$

Who ordered that?

$$H = \sum_{ij} S_i^x S_j^x + S_i^y S_j^y - 0.5 \sum_{ij} S_i^z S_j^z$$

$$|+\rangle \equiv \frac{1}{\sqrt{3}}\Big(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\Big)$$

$$|-\rangle \equiv \frac{1}{\sqrt{3}}\Big(|\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle\Big)$$

$$H_{\text{tri}} = -\frac{3J}{8}\sum_{i=1}^{6} |i\rangle\langle i| + \frac{9J}{8}(|+\rangle\langle +| + |-\rangle\langle -|)$$

$$-\frac{3J}{8}(1 - |+\rangle\langle +| - |-\rangle\langle -|)$$

$$|+\rangle \equiv \frac{1}{\sqrt{3}}\Big(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\Big)$$
$$|-\rangle \equiv \frac{1}{\sqrt{3}}\Big(|\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle\Big)$$

$$-\frac{3J}{8} + \frac{3J}{2}\left(|+\rangle\langle+| + |-\rangle\langle-|\right)$$

Projectors

Constant

Positive coefficient

We want to **minimize the energy** by zeroing out the projectors

Frustration Free!

Many Triangles

$$H = \sum_{\text{tri}} H_{\text{tri}} = \frac{3}{2} \sum_{\text{tri}} P_{\text{tri}} - \frac{3}{8} N_{\text{tri}}$$

$$P_{\text{tri}} \equiv |+\rangle\langle+| + |-\rangle\langle-|$$

NOTE: projectors on triangles **DO NOT** commute with each other!!!

So an arbitrary eigenstate on one triangle need not be COMPATIBLE with other triangles

$$H = \sum_{\text{tri}} H_{\text{tri}} = \frac{3}{2} \sum_{\text{tri}} P_{\text{tri}} - \frac{3}{8} N_{\text{tri}}$$

$$P_{\text{tri}} \equiv |+\rangle\langle+| + |-\rangle\langle-|$$

$$|a\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle + |\downarrow\rangle\Big)$$

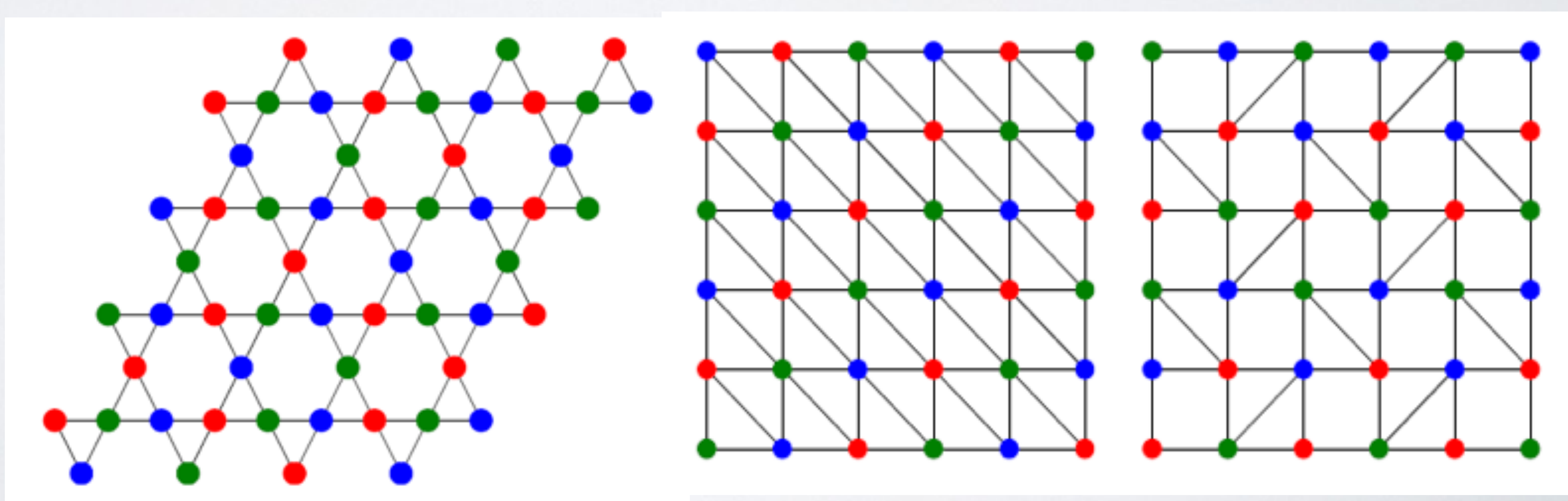
$$|b\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle + \omega|\downarrow\rangle\Big)$$

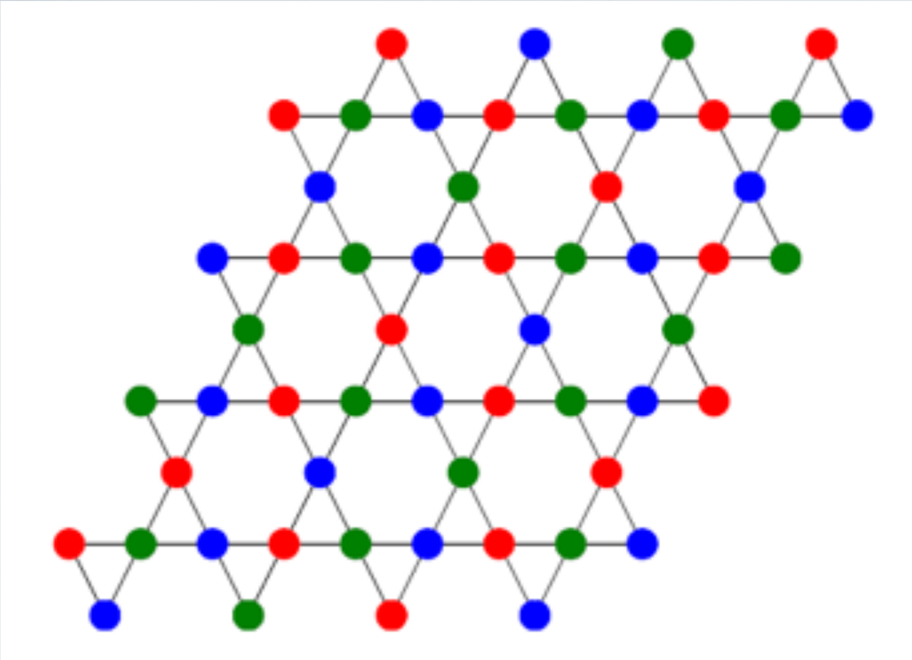
$$|c\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle + \omega^2|\downarrow\rangle\Big)$$

We want projector to annihilate our proposed solution

$$|\psi\rangle \equiv \prod_s \otimes |C_s\rangle_s$$

As long as we have ONLY one color per triangle the tensor product of all colors is an EXACT eigenstate





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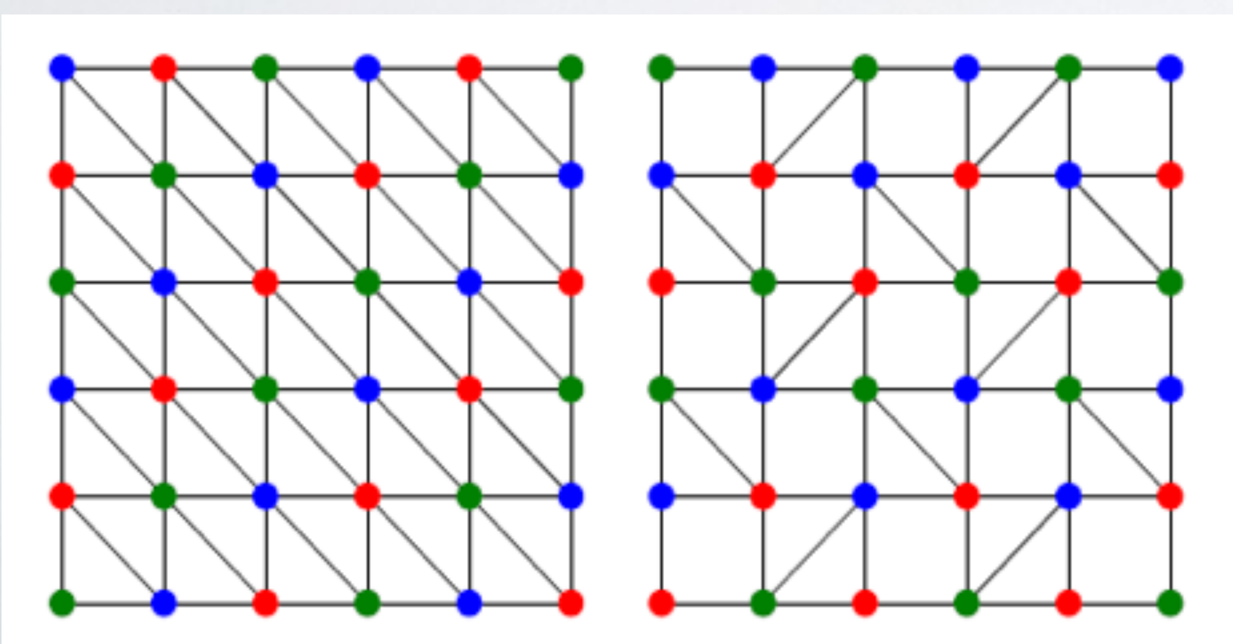
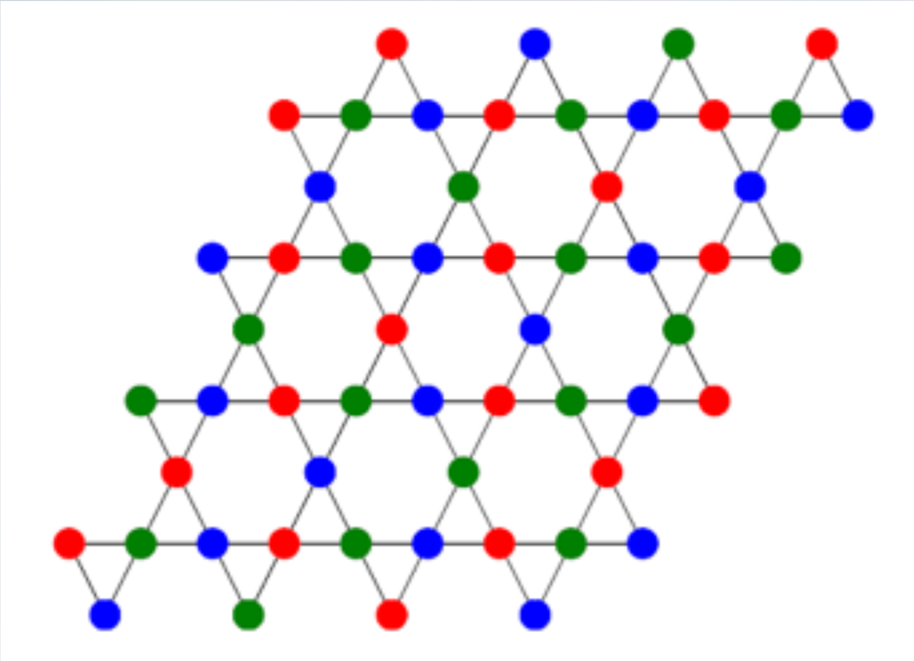
An exponential number of colorings! 1.208^N (from Baxter)

$$|\psi\rangle \equiv \prod_s \otimes |C_s\rangle_s$$

Only one (or two) colorings.





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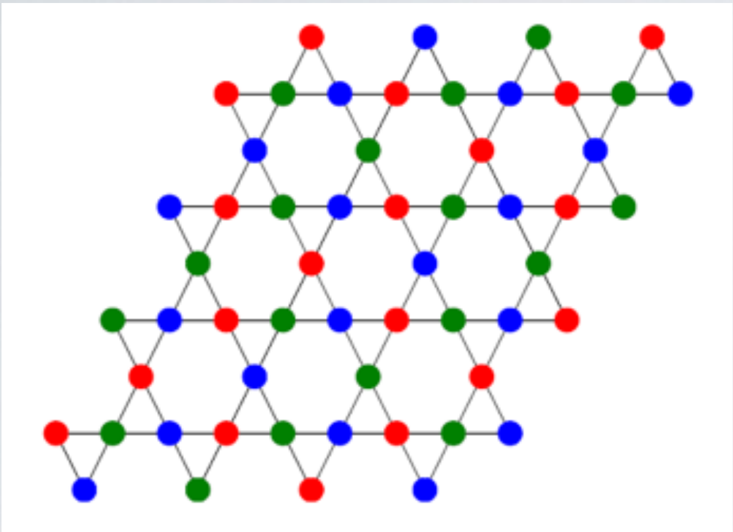
An exponential number of colorings!

$$|\psi\rangle \equiv \prod_s \otimes |C_s\rangle_s$$

But this mixes Sz sectors, (particle number in boson language)

Eigenstates in a fixed Sz sector

Lattice	Ising configs	Colorings
2x2x3	924	8
3x2x3	48620	16
4x2x3	2.7 million	32



$$|\psi\rangle \equiv \prod_s \otimes |C_s\rangle_s$$

But this mixes Sz sectors, (particle number in boson language)

⇩ Project to definite sector

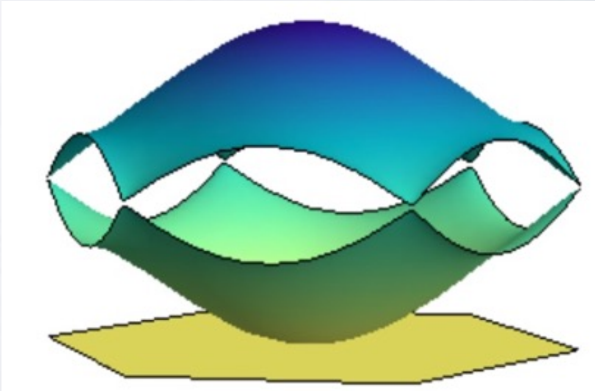
$$|\psi^C\rangle \equiv P_{S_z}\left(\prod_{\text{valid}} \otimes |C_s\rangle\right)$$

These are all still eigenstates!

Modes may be linearly dependent. Their rank may be less then the number of colorings.

Additional Subtelty: These are not always all the eigenstates.

The "one-boson" particle number sector reproduces the known flat band.

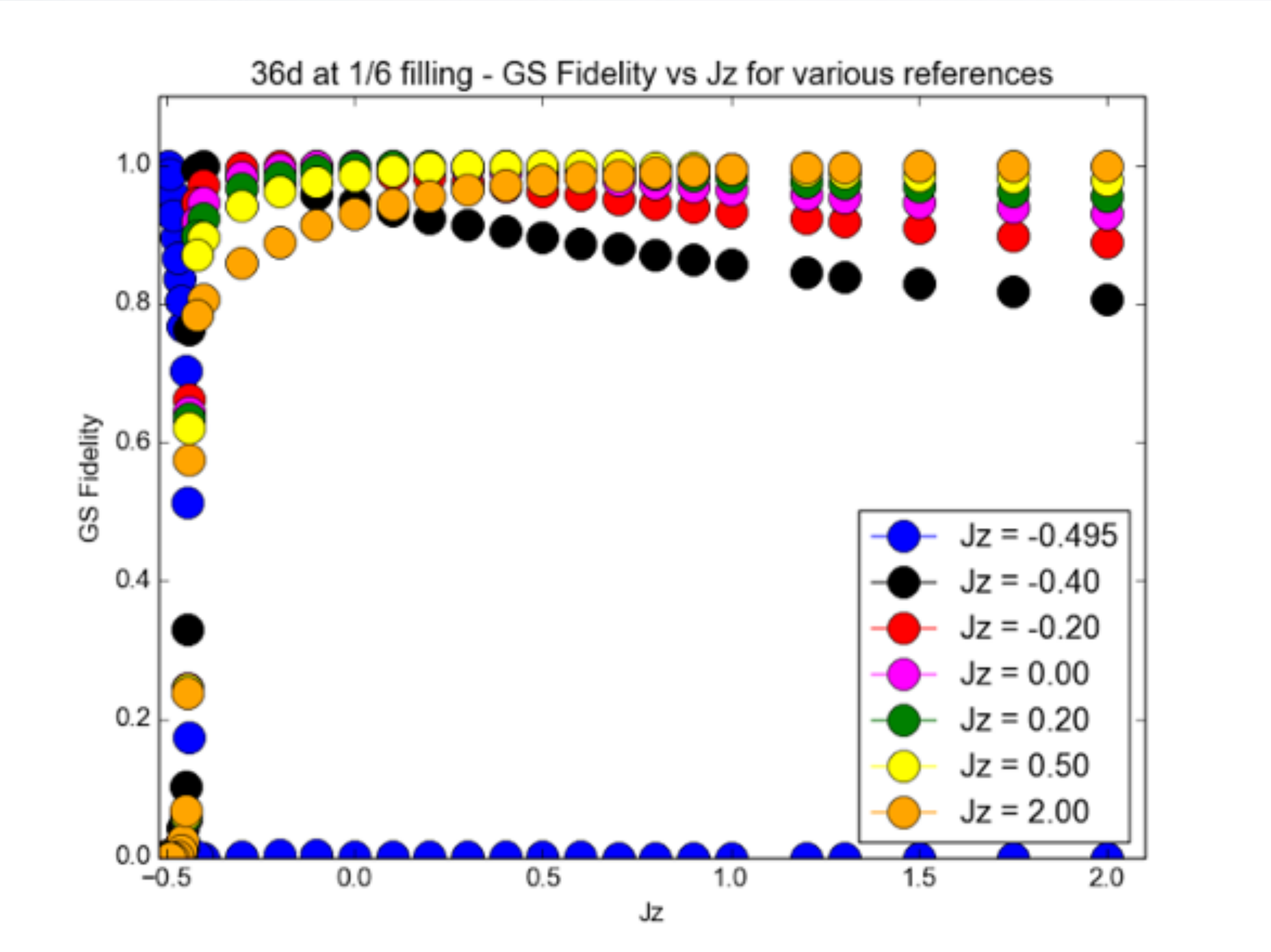


How should we move away from this special point?

$$H = H^* + xH'$$

$$H' \equiv \sum_{\langle i,j \rangle} S_i^z S_j^z$$

$$x \equiv (J_z + 1/2)$$



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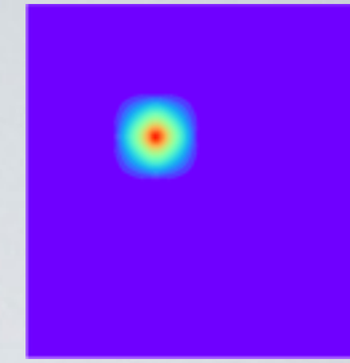

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$$H' \equiv \sum_{\langle i,j \rangle} S_i^z S_j^z$$

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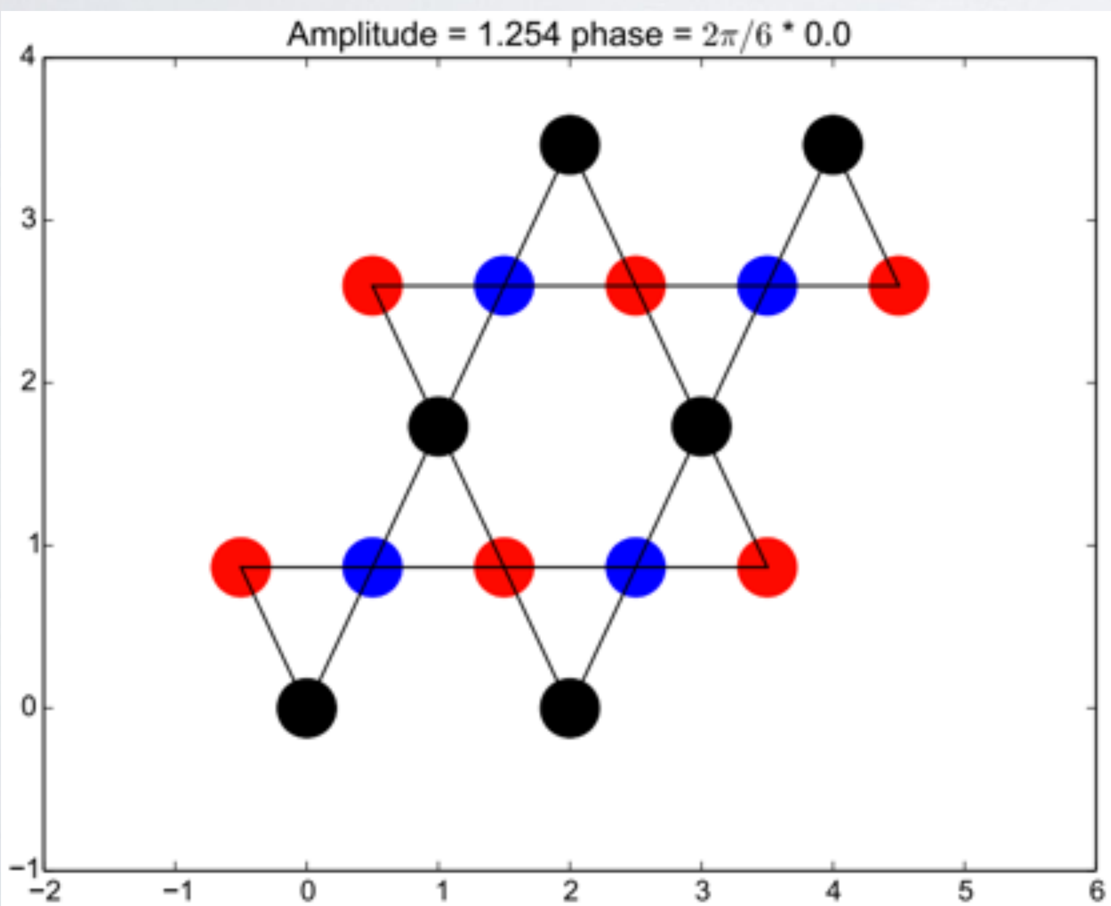
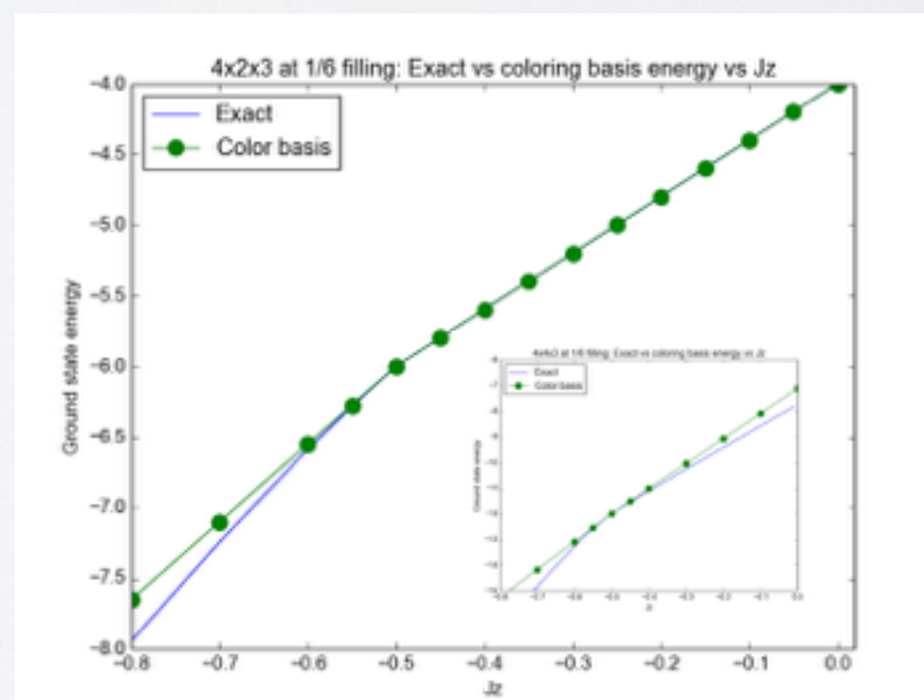
What state survives from the degenerate manifold?

Subtle point: Not sure its gapped.
Ignore that and march forward anyway…



At 1/6 filling, all states have $0, \omega, \omega^2$ phases

All non-zero amplitudes the same but the chiral state.



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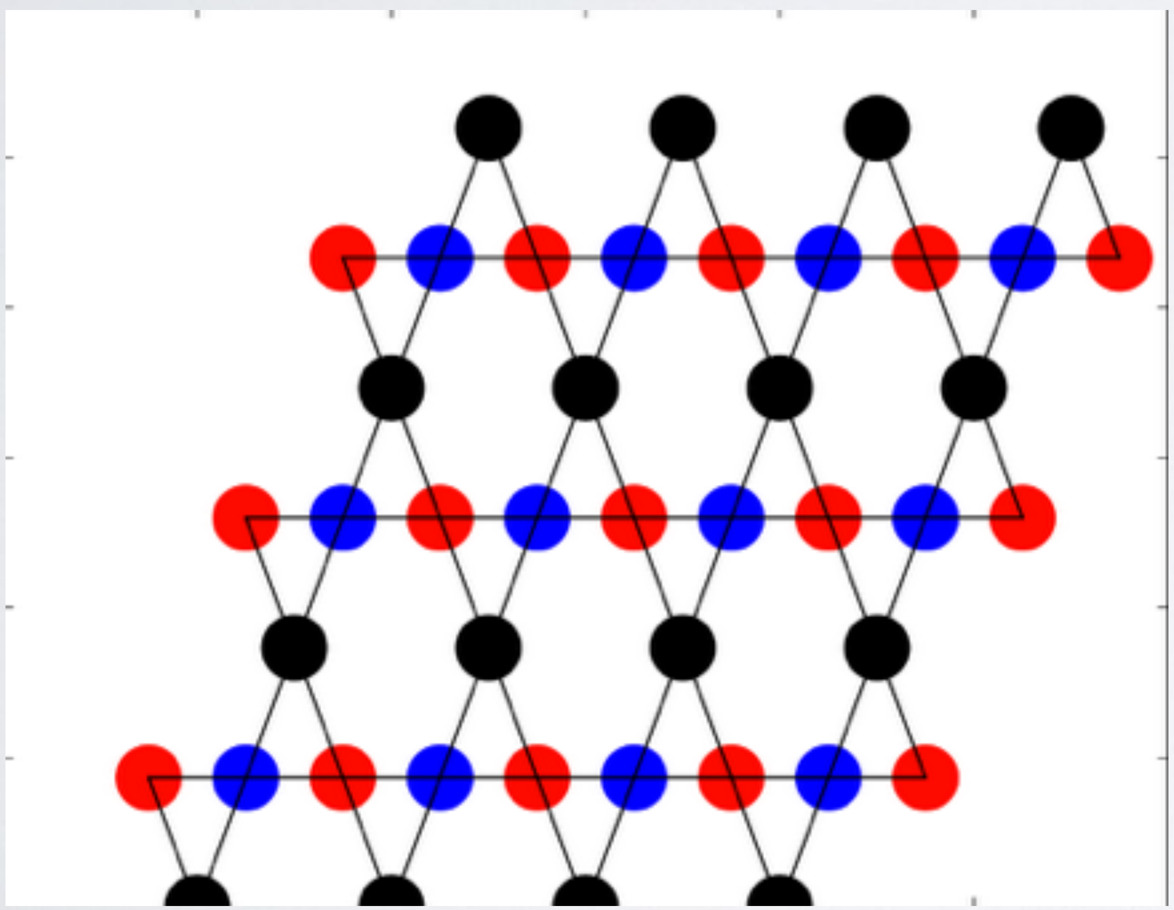

At 1/2 filling, all states have +/- phase and zero or the same amplitude.

Conclusions

$$H = \sum_{ij} S_i^x S_j^x + S_i^y S_j^y - 0.5 \sum_{ij} S_i^z S_j^z$$

Paste together triangles

Three color it so each triangle has one red, one blue, one green

Project this into a Sz sector

This is a ground state.

On kagome this is extensively degenerate