UNIVERSITY OF ILLINOIS
AT URBANA-CHAMPAIGN

Basic Error Analysis

Physics 403

Eugene V Colla
D. Hertzog
A. Bezryadin

Outline of the lecture

- **Errors and uncertainties**
- **The reading error**
- **Accuracy and precession**
- **Systematic and statistical errors**
- **Fitting errors**
- **Presentation of the results**
- **Heisenberg limit precision measurements**

Introduction

- Uncertainties exist in all experiments
- The final goal of any experiment is to obtain *reproducible* results. Knowing errors and uncertainties is an essential part for ensuring reproducibility.
- To know the uncertainties we use two approaches:

(1) Repeat each measurement many times and determine how well the result reproduces itself. If the results are different then there are **statistical errors**.

(2) Measure the quantity of interest using a different method. The results, if correct, are independent of the measurement technique. If the results are different then there are **systematic errors** in one of the methods or in both.

(3) Presenting the result of your experiment: Use the right number of significant digits, in agreement with the estimated uncertainty.

Errors (uncertainties)

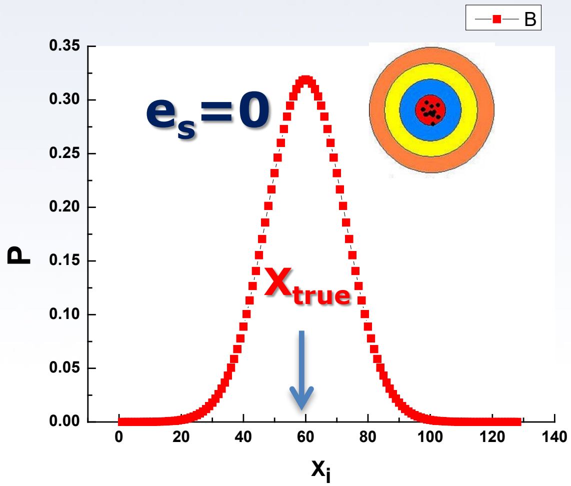
Result of measurement

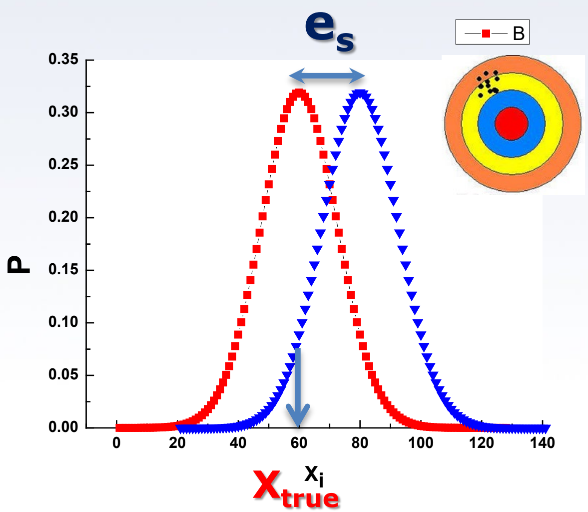
Systematic error

$$X_{meas} = X_{true} + e_s + e_r$$

Correct value

Random error





Systematic vs. Statistical Uncertainties

- Systematic uncertainty
  - Uncertainties associated with imperfect knowledge of measurement **apparatus**, other physical **quantities** needed for the measurement, or the physical **model** used to interpret the data.
  - Generally correlated between measurements. Cannot be reduced by multiple measurements.
  - Better calibration, or measurements employing different techniques or methods can reduce the uncertainty.
- Statistical Uncertainty
  - Uncertainties due to stochastic fluctuations
  - Generally there is no correlation between successive measurements.
  - Multiple measurements can be used to reduce this uncertainty.

The Difference Between Systematic & Random Errors

- Random error describes errors that fluctuate due to the unpredictability or uncertainty inherent in your measuring process, or the variation in the quantity you're trying to measure. Such errors can be reduced by repeating the measurement and averaging the results.

- A systematic error is one that results from a persistent issue and leads to a consistent error in your measurements. For example, if your measuring tape has been stretched out, your results will always be lower than the true value. Similarly, if you're using scales that haven't been set to zero beforehand, there will be a systematic error resulting from the mistake in the calibration. Such errors cannot be reduced simply by repeating the measurement and averaging the results. Such errors can be reduced by analyzing the instrument(s) used for the measurement and by using different instruments.

Definitions (NIST)

The standard uncertainty σ of a measurement result x is the estimated standard deviation of x.

The relative standard uncertainty σ_r of a measurement result x is defined by $\sigma_r = \sigma / |x|$, where x is not equal to 0.

In statistics, the standard deviation (**SD, also represented by the Greek letter sigma σ**) is a measure that is used to quantify the amount of variation or dispersion of a set of data values. *A low standard deviation indicates that the data points tend to be close to the mean value of the set ($\mu = \langle x_i \rangle$), while a high standard deviation indicates that the data points are spread out over a wider range of values.*

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2}, \quad \text{where} \quad \mu = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Meaning

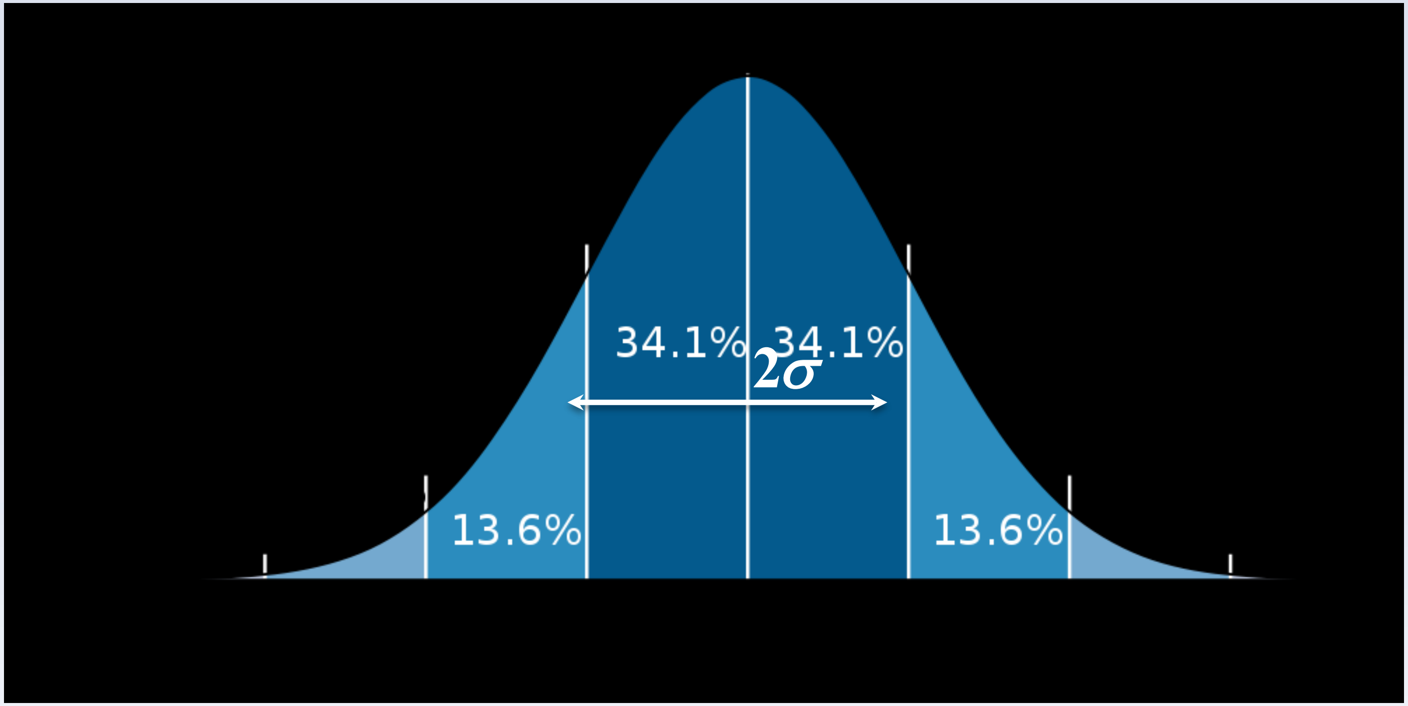
Meaning of uncertainty:

Assume the distribution of the measurement results is normal (Gaussian).
If the result of a measurement is **x**, and the standard deviation is σ, then the interval **x** – σ to **x** + σ is expected to encompass approximately 68 % of the measurement results (if the measurement is repeated again and again).

Let us **X** is the true value (never known exactly) and **x** is the measured value. The probability that the true value **X** is greater than **x** - σ, and is less than **x** + σ is estimated as 68%.

This statement is commonly written as **X**= **x** ± σ.

Normal (Gaussian) distribution



$$P_n(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$

The interval representing two standard deviations contains 95.4% of all possible true values.
Confidence interval <x> ± 3σ contains 99.7% of possible outcomes.

Notations

Use of concise notation:

If, for example, v = 1 234.567 89 m/s and Δv = 0.000 11 m/s, where m/s is the unit of v, then v = (1 234.567 89 ± 0.000 11) m/s.

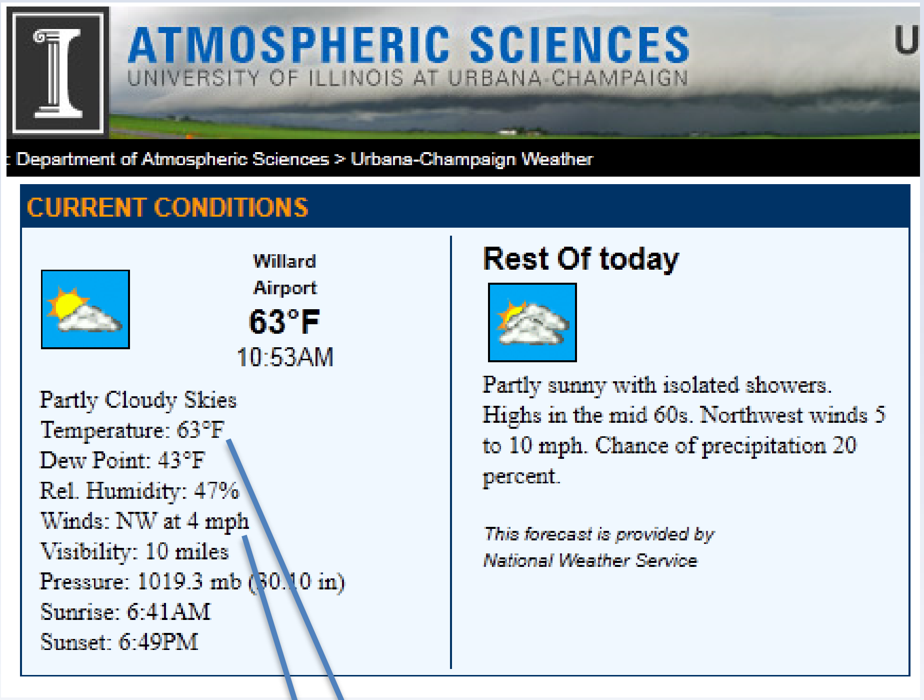
A more concise form of this expression, and one that is used sometimes, is v = 1 234.567 89(11) m/s, where it understood that the number in parentheses is the numerical value of the standard uncertainty referred to the corresponding last digits of the quoted result.

Examples of results which do not make sense (too many digits):

v = (1234.5678934534940945 ± 0.011) m/s

or v = (1234.56 ± 2) m/s

Significant digits



$T = 63°\text{F} \pm ?$ → **Best guess** $\Delta T \sim 0.5°\text{F}$

Wind speed 4mph$\pm ?$ → **Best guess** $\pm 0.5 mph$

If they say T=63.32456 F, that would be wrong
since it is not possible to predict or even measure the temperature at our campus with such high precision.

It is important to know uncertainties in science

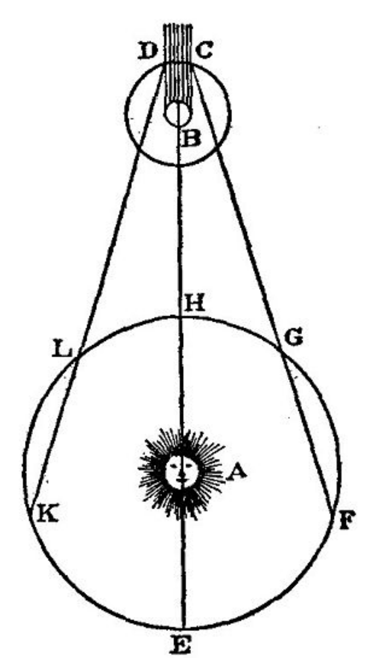


FIG. 70.

Measurement of the speed of the light

1675 Ole Roemer: 220,000 Km/sec


Ole Christensen Rømer
1644-1710

Does it make sense?
What is missing?

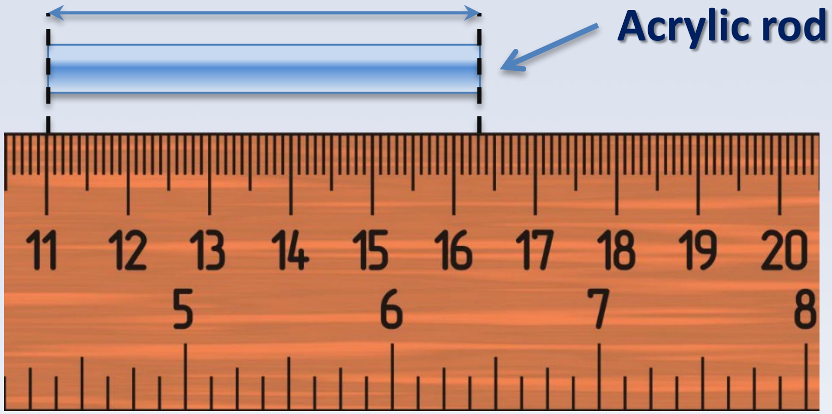
Maxwell's theory prediction:
The speed of light does not depend on the light wavelength, frequency or color. It is a universal constant.
NIST Bolder Colorado c = 299,792,456.2±1.1 m/s.

Reading error

$\Delta L \cong 0.5mm$

L=53mm±ΔL(?)

Acrylic rod



$\Delta L \cong 0.03mm$

How far we have to go in reducing the reading error?

Use a simple ruler if you do not care about accuracy better than 1mm

Otherwise you need to use digital calipers

Probably the natural limit of accuracy can be due to length uncertainty because of temperature expansion. For 53mm $\Delta L \cong 0.012mm/K$

Reading Error = $\pm\frac{1}{2}$ (least count or minimum gradation).

Reading error. Digital meters.

Fluke 8845A multimeter

Example Vdc (reading)=0.85V

$$\Delta V = 0.85 \times \left(1.8 \times 10^{-5}\right) \sim 15\mu V$$

8846A Accuracy

Accuracy is given as ± (% measurement + % of range)

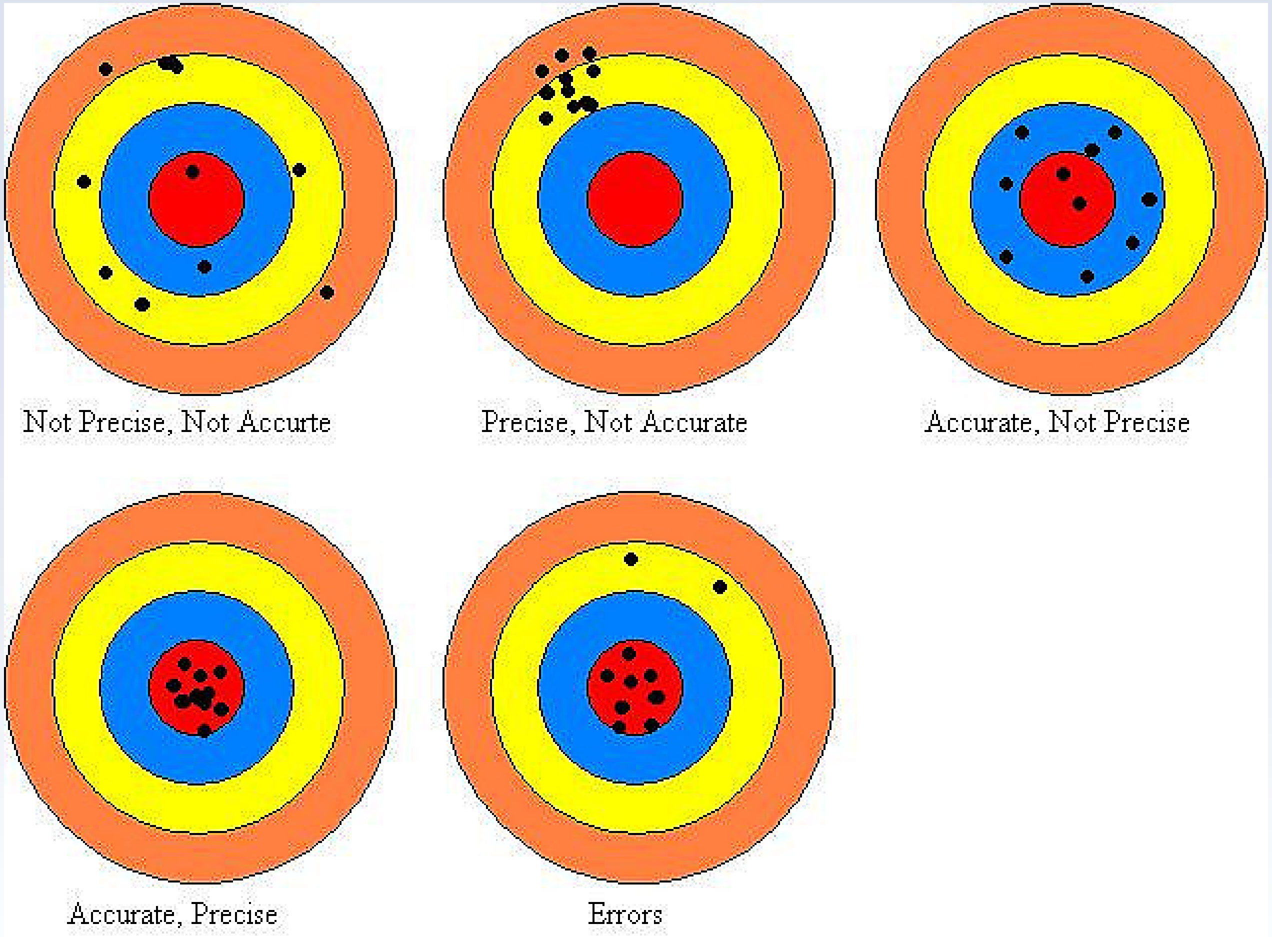
Range	24 Hour (23 ±1 °C)	90 Days (23 ±5 °C)	1 Year (23 ±5 °C)	Temperature Coefficient/ °C Outside 18 to 28 °C
100 mV	0.0025 + 0.003	0.0025 + 0.0035	0.0037 + 0.0035	0.0005 + 0.0005
1 V	0.0018 + 0.0006	0.0018 + 0.0007	0.0025 + 0.0007	0.0005 + 0.0001
10 V	0.0013 + 0.0004	0.0018 + 0.0005	0.0024 + 0.0005	0.0005 + 0.0001
100 V	0.0018 + 0.0006	0.0027 + 0.0006	0.0038 + 0.0006	0.0005 + 0.0001
1000 V	0.0018 + 0.0006	0.0031 + 0.001	0.0041 + 0.001	0.0005 + 0.0001

Accuracy and precession

The accuracy of an experiment is a measure of how close the result of the experiment comes to the true value

Precision refers to how closely individual measurements agree with each other

Accuracy and precession



Systematic and random errors

- ***Systematic Error*: reproducible inaccuracy introduced by faulty equipment, calibration, technique, model, drifts.**

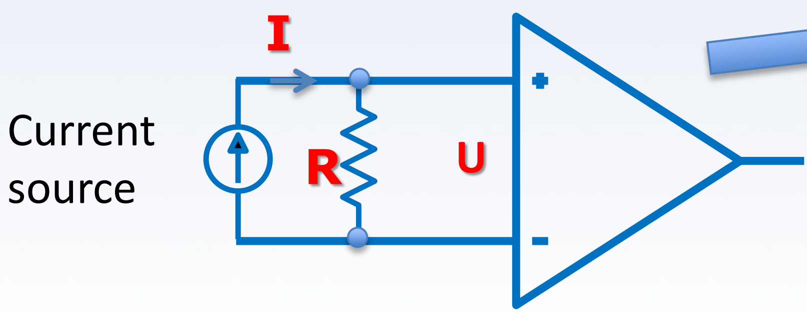
- ***Random errors*: Indefiniteness of results due to finite precision of experiment. Errors can be reduced be repeating the measurement and averaging. These errors can be caused by thermal motion of molecules and electrons in the apparatus.**

Philip R. Bevington "Data Reduction and Error Analysis for the Physical sciences", McGraw-Hill, 1969

Systematic errors

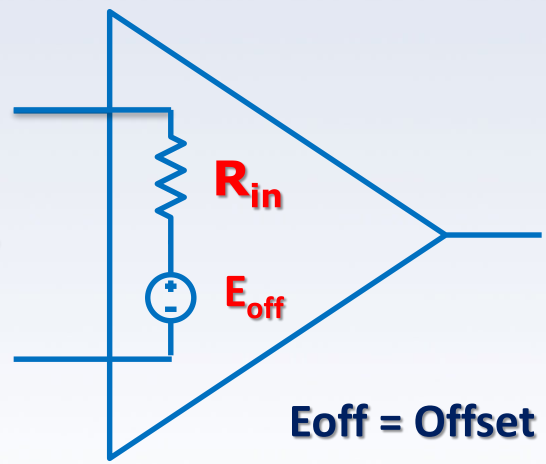
Sources of systematic errors: **poor calibration of the equipment, changes of environmental conditions, imperfect method of observation, drift and some offset in readings etc.**

Example #1: measuring of the DC voltage



expectation

U=R*I



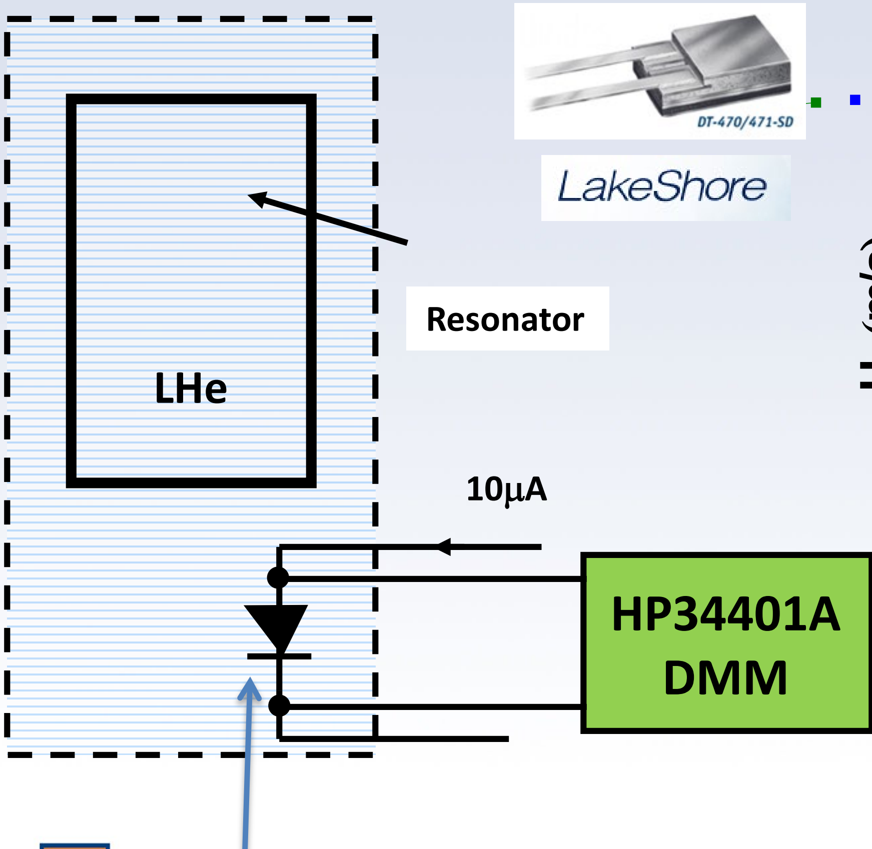
Eoff = Offset Votlage

actual result

U = RI + Eoff

Systematic errors

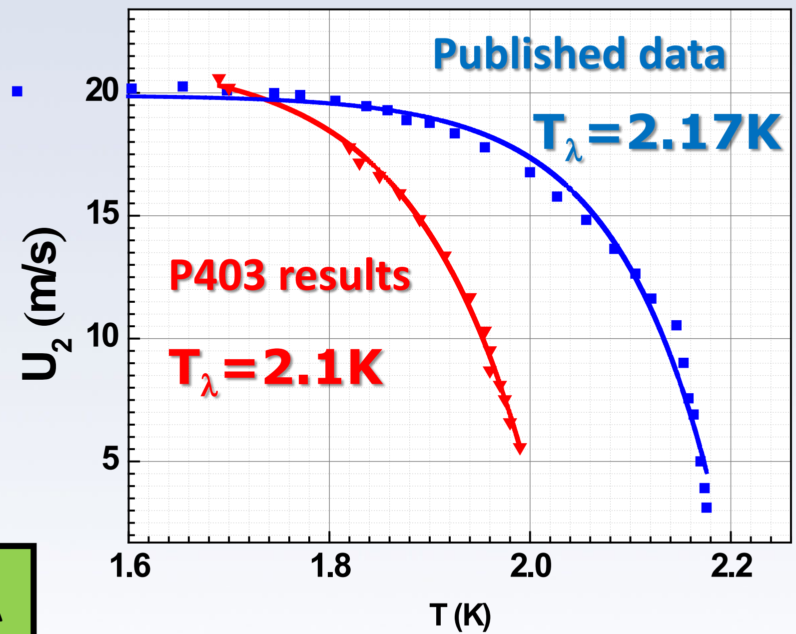
Example #3: poor calibration



LakeShore

Temperature sensor

Measuring of the speed of the second sound in superfluid He4



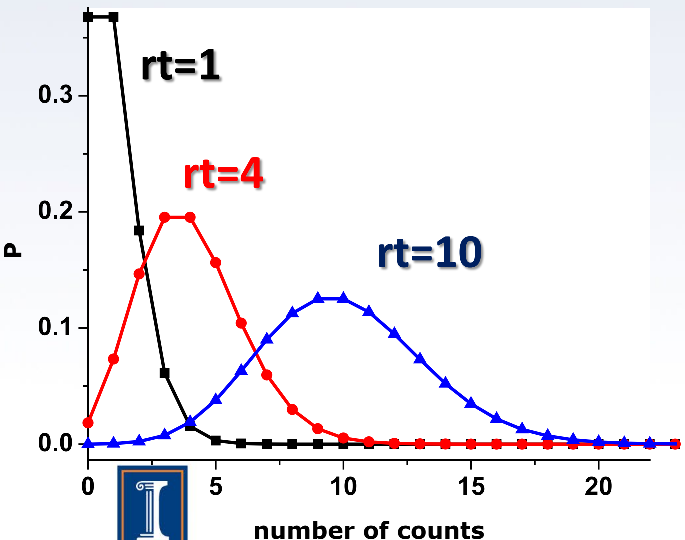
Random errors. Poisson distribution

Siméon Denis Poisson
(1781-1840)

$$P_n(t) = \frac{(rt)^n}{n!} e^{-rt} \qquad n = 0,1,2,\ldots$$

r: decay rate [counts/s] t: time interval [s]

➔ $P_n(rt)$: Probability to have n decays in time interval t



A statistical process is described through a Poisson Distribution if:

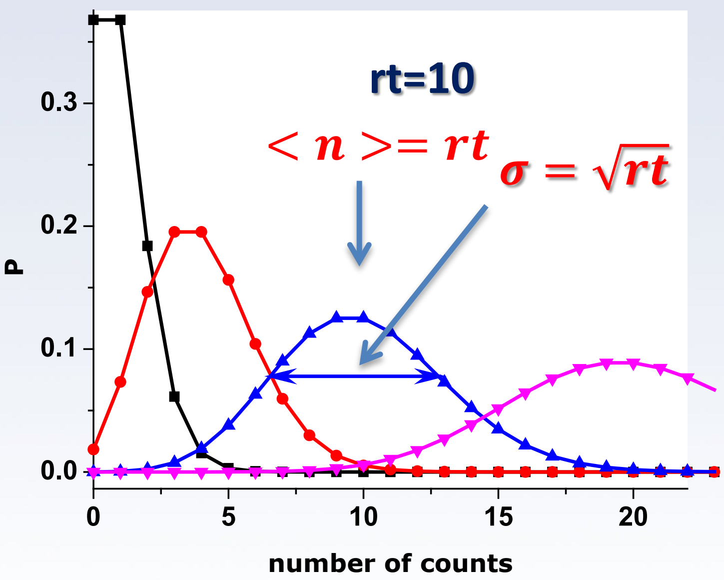
- **random process** → for a given nucleus probability for a decay to occur is the same in each time interval.
- **universal probability** → the probability to decay in a given time interval is same for all nuclei.
- **no correlation between two instances** (the decay of on nucleus does not change the probability for a second nucleus to decay.

Poisson distribution

$$P_n(t) = \frac{(rt)^n}{n!} e^{-rt} \quad n = 0, 1, 2, \ldots$$

r: decay rate [counts/s] t: time interval [s]

➔ $P_n(rt)$: Probability to have n decays in time interval t



Properties of the Poisson distribution:

$$\sum_{n=0}^{\infty} P_n(rt) = 1$$, probabilities sum to 1

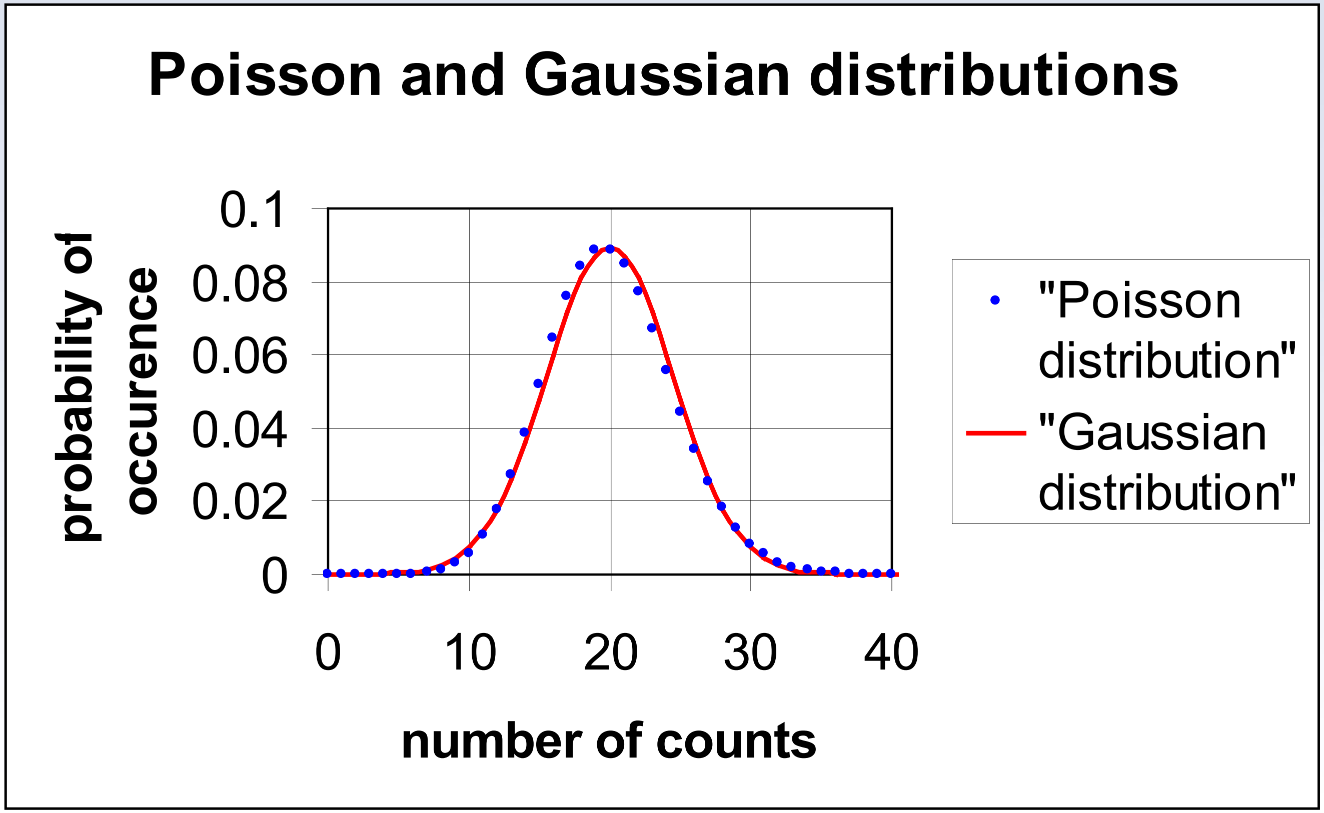
$$< n > = \sum_{n=0}^{\infty} n \cdot P_n(rt) = rt$$, the mean

$$\sigma = \sqrt{\sum_{n=0}^{\infty} (n - < n >)^2 P_n(rt)} = \sqrt{rt}$$, standard deviation

illinois.edu

Poisson distribution at large *rt*

$$P_n(t) = \frac{(rt)^n}{n!} e^{-rt} \quad n = 0,1,2,\ldots$$

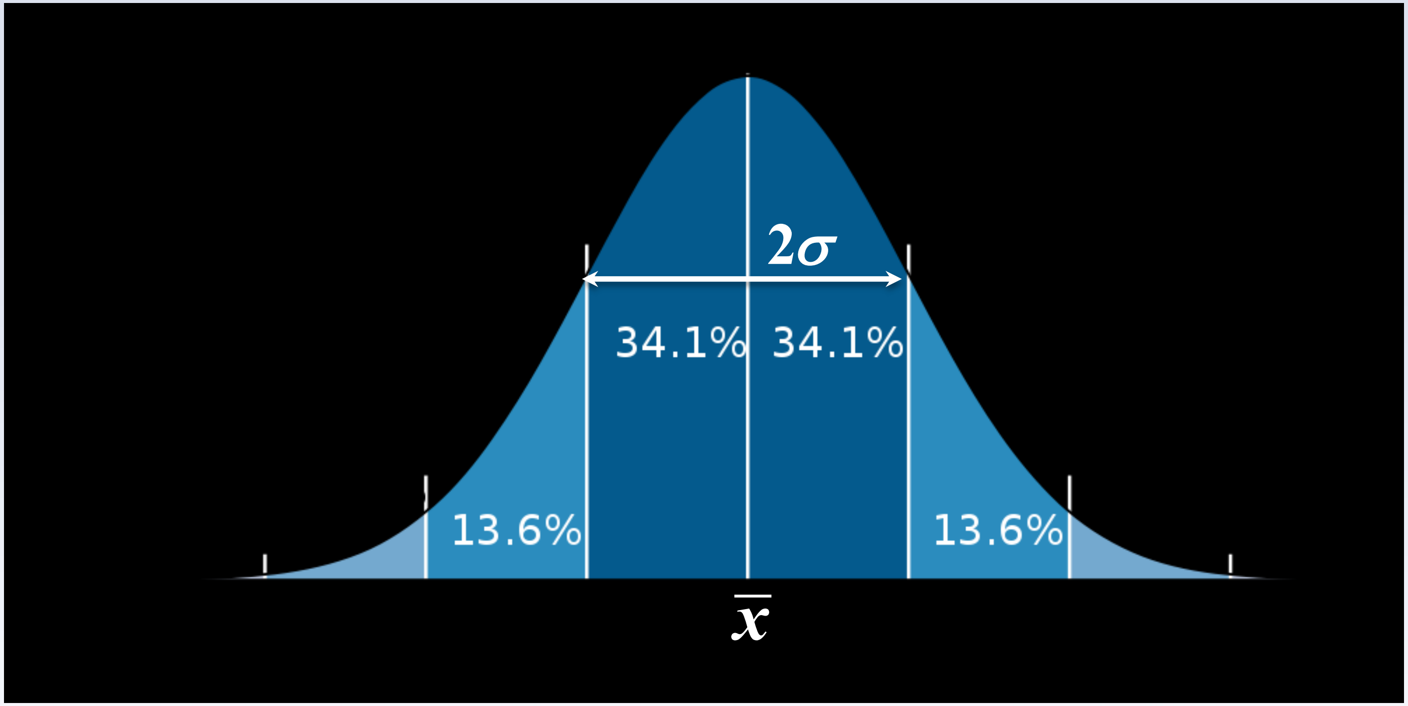


Carl Friedrich Gauss
(1777–1855)

$$P_n(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$

Gaussian distribution: continuous

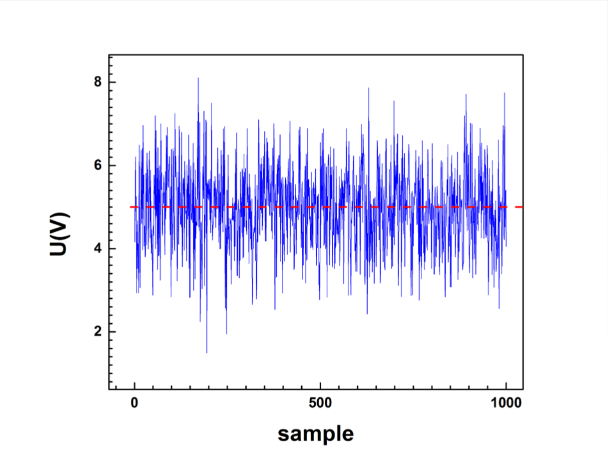
Normal (Gaussian) distribution



$$P_n(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$

Measurement in presence of noise

Source of noisy signal

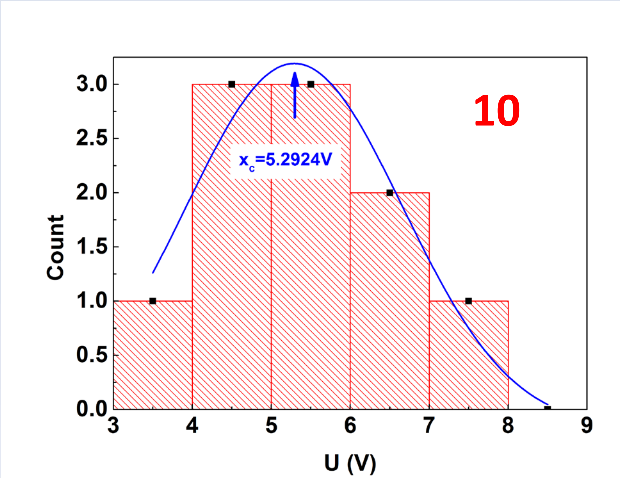


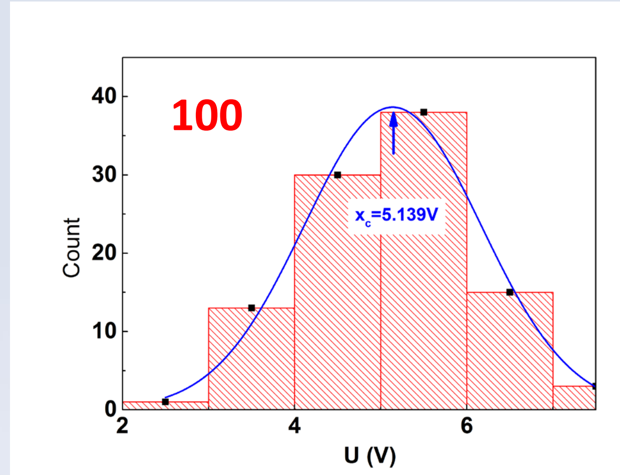
4.89855
5.25111
2.93382
4.31753
4.67903
3.52626
4.12001
2.93411

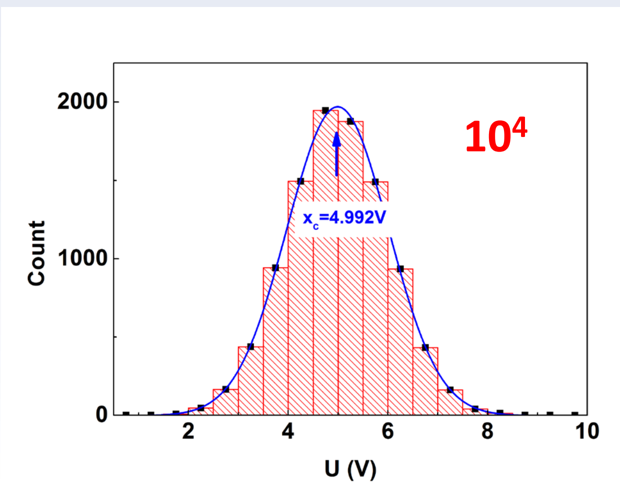
Expected value 5V

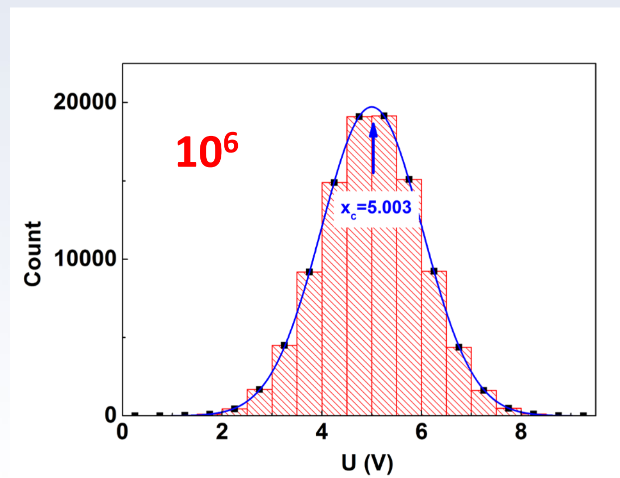
Actual measured values

Measurement in presence of noise









Error in the mean is given as $\frac{\sigma 0}{\sqrt{N}}$ (This is called standard quantum limit or the shot noise limit)

Heisenberg limit measurments

According to Heisenberg uncertainty,

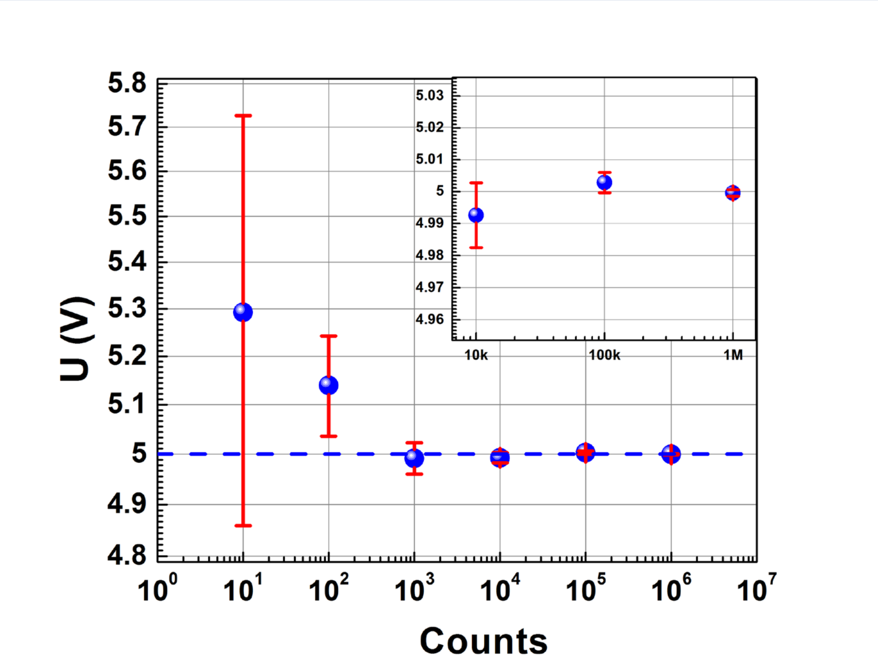
the ultimate precision of the energy measurement is $\Delta E \sim \frac{\hbar}{t}$

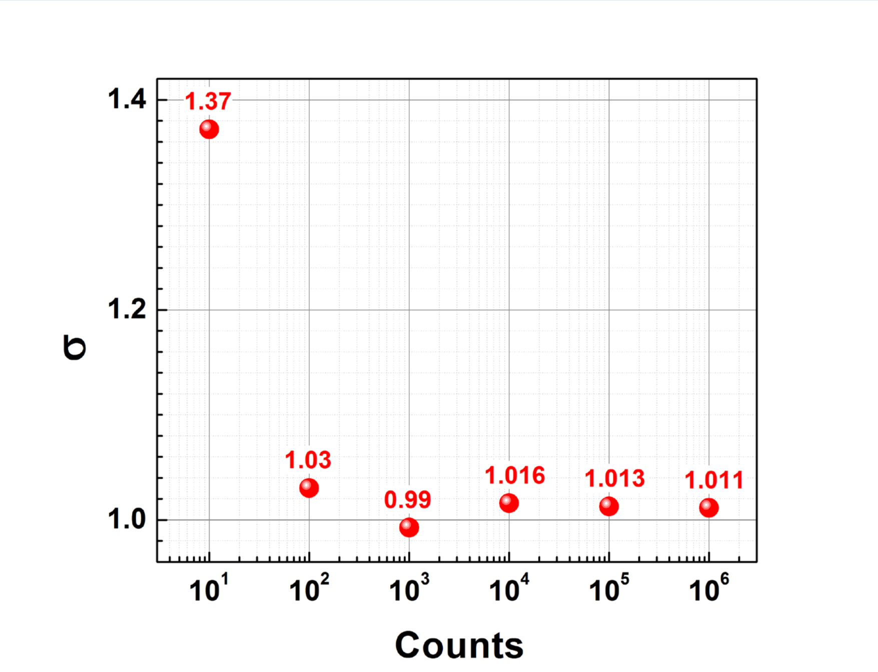
If N is the number of measurements performed then $t=N*t_1$, where t_1 is the time needed to perform one measurement.

Thus the precision can be as good as $\Delta E \sim \frac{\hbar}{t_1}\frac{1}{N}$

To achieve this high precision one has to use a quantum system, such as a **qubit**.

Measurement in presence of noise





Result → $U = x_c \pm \frac{\sigma}{\sqrt{N}}$

σ - standard deviation
N – number of samples

For N=10^6 U=4.999±0.001 0.02% accuracy

Measurement in presence of noise

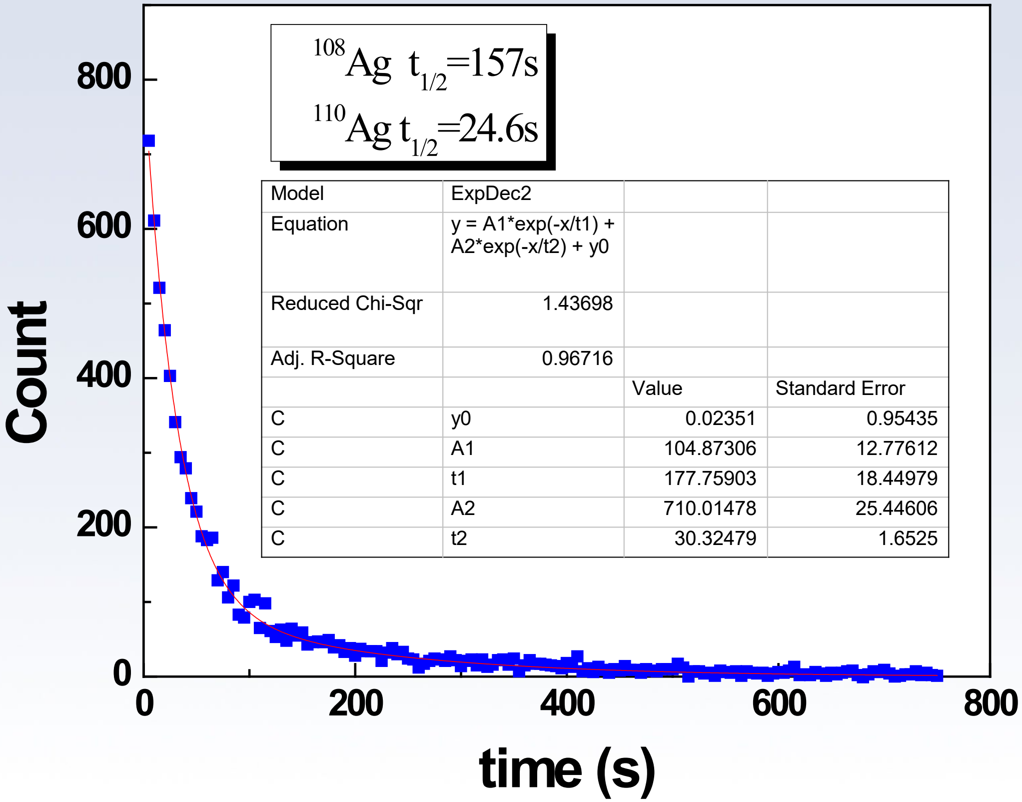
The standard error equals the standard deviation divided by the square root of the sample size (=number of measurements).
In other words, the standard error of the mean is a measure of the dispersion of sample means around the population mean.

Result → $$U = x_c \pm \frac{\sigma}{\sqrt{N}}$$

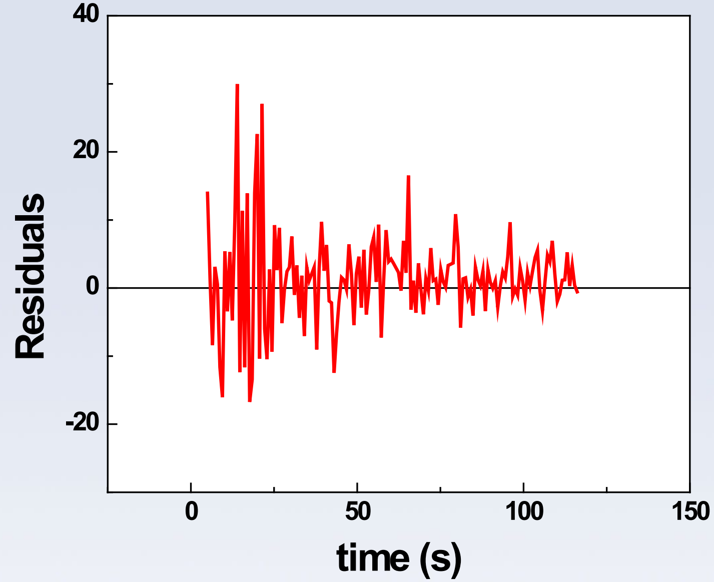
σ - standard deviation
N – number of samples

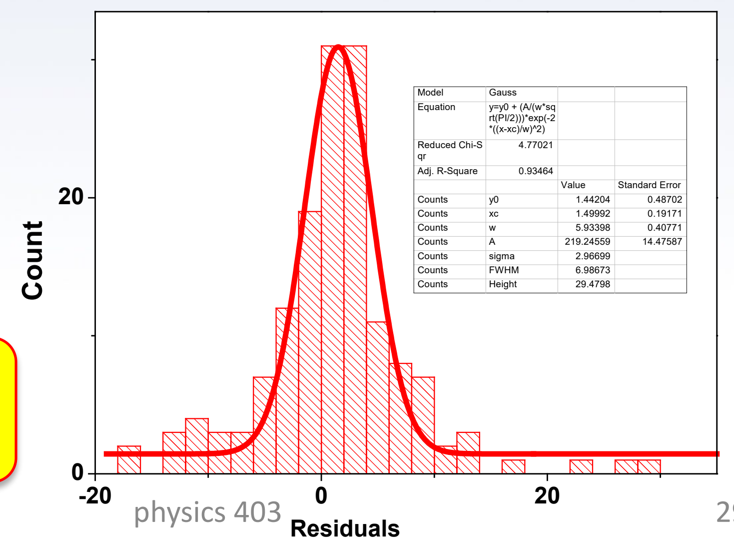
Fitting errors

Ag β decay



Model	ExpDec2		
Equation	y = A1*exp(-x/t1) + A2*exp(-x/t2) + y0		
Reduced Chi-Sqr	1.43698		
Adj. R-Square	0.96716		
		Value	Standard Error
C	y0	0.02351	0.95435
C	A1	104.87306	12.77612
C	t1	177.75903	18.44979
C	A2	710.01478	25.44606
C	t2	30.32479	1.6525



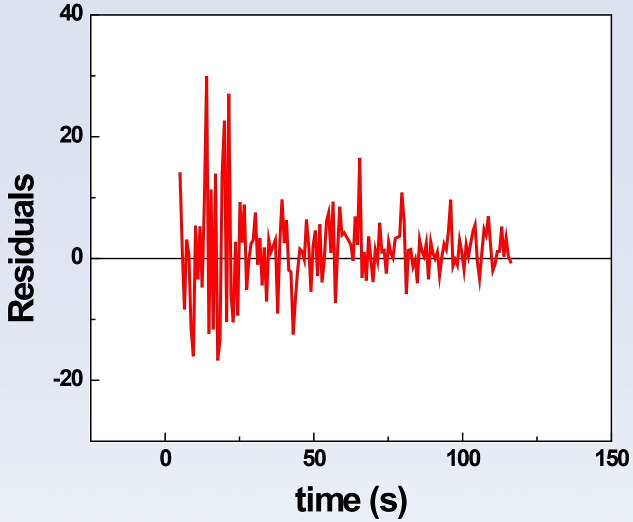


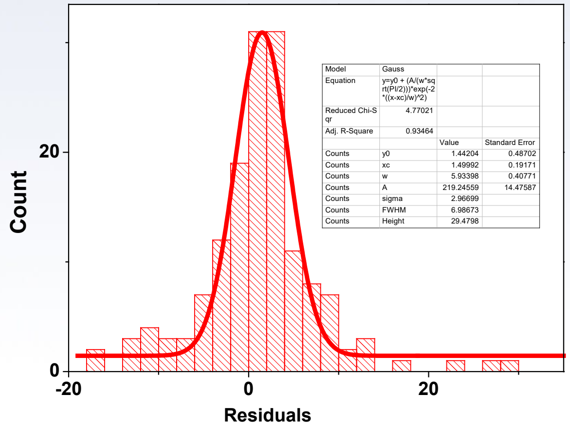
Model	Gauss		
Equation	y=y0 + (A/(w*sqrt(PI/2)))*exp(-2*((x-xc)/w)^2)		
Reduced Chi-Sqr	4.77021		
Adj. R-Square	0.93464		
		Value	Standard Error
Counts	y0	1.44204	0.48702
Counts	xc	1.49992	0.19171
Counts	w	5.93398	0.40771
Counts	A	219.24559	14.47587
Counts	sigma	2.96699	
Counts	FWHM	6.98673	
Counts	Height	29.4798	

$$y = A1 \bullet \exp\left(\frac{-t}{t_1}\right) + A2 \bullet \exp\left(\frac{-t}{t_2}\right) + y_0$$

Fitting. Analysis of the residuals

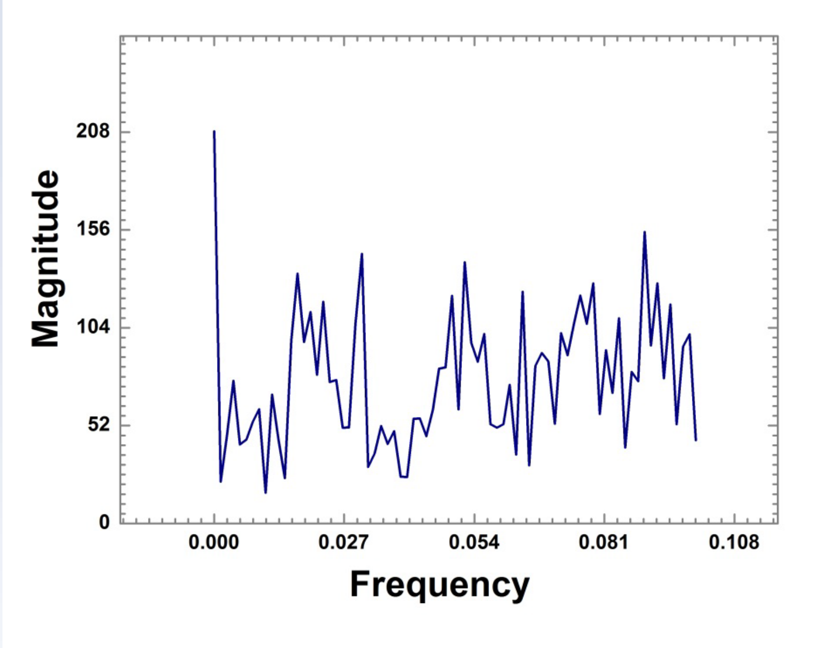
Ag β decay





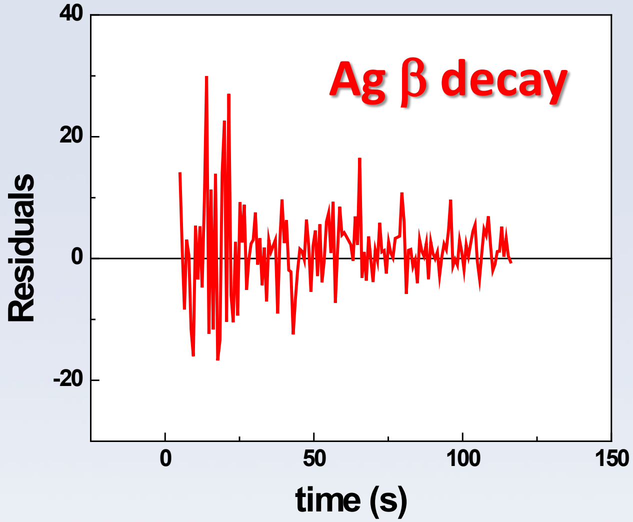
Model	Gauss		
Equation	y=y0 + (A/(w*sqrt(PI/2)))*exp(-2*((x-xc)/w)^2)		
Reduced Chi-Sqr	4.77021		
Adj. R-Square	0.93464		
		Value	Standard Error
Counts	y0	1.44204	0.48702
Counts	xc	1.49992	0.19171
Counts	w	5.93398	0.40771
Counts	A	219.24559	14.47587
Counts	sigma	2.96699	
Counts	FWHM	6.98673	
Counts	Height	29.4798	

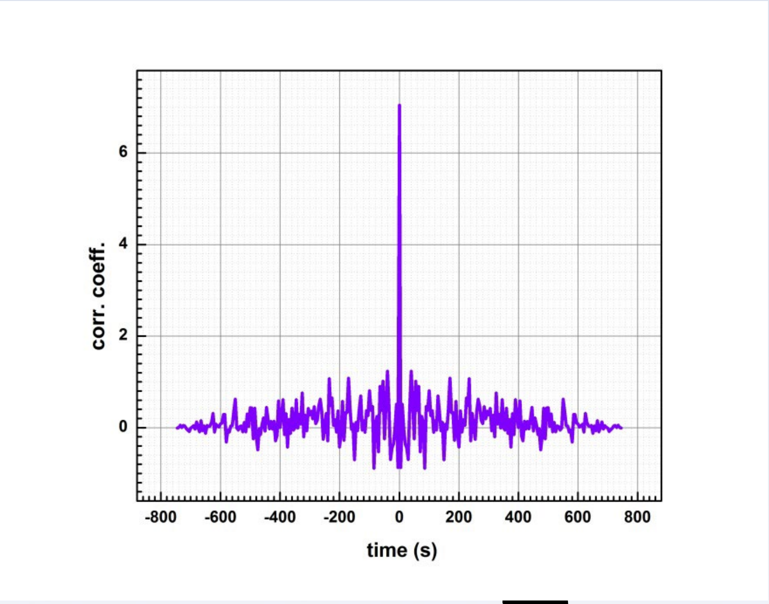
Test 1. Fourier analysis

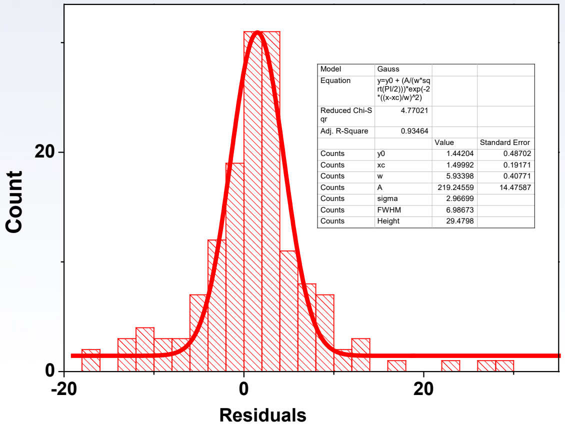


No pronounced frequencies found

Fitting. Analysis of the residuals







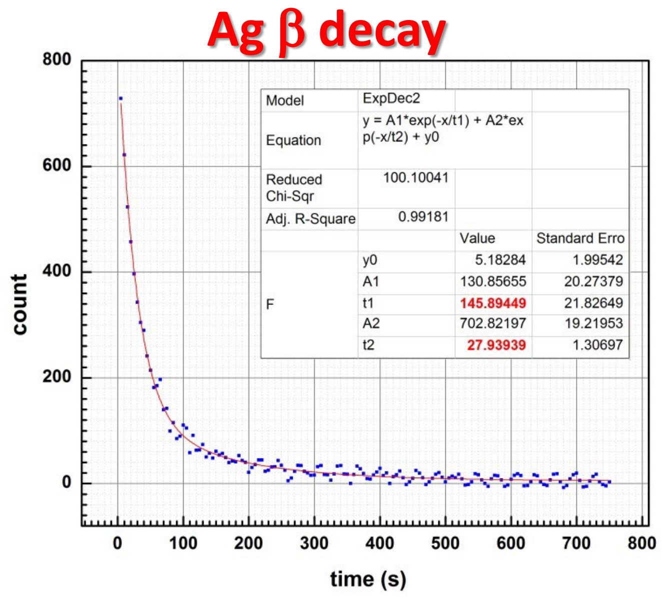
Correlation function

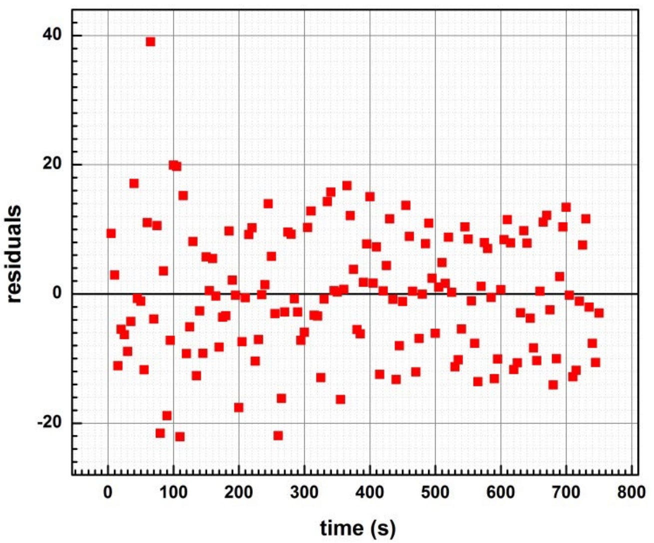
$$y(m) = \sum_{n=0} f(n)g(n-m)$$

autocorrelation function

$$y(m) = \sum_{n=0}^{M-1} f(n)f(n-m)$$

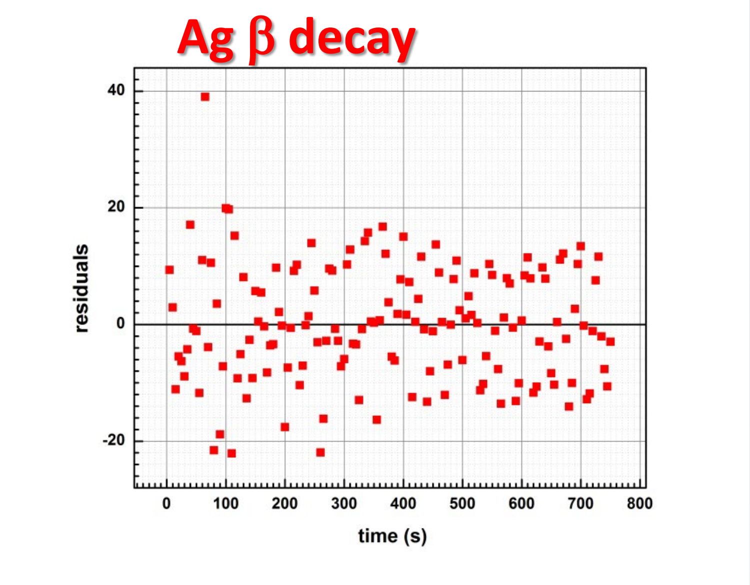
Fitting. Analysis of the residuals. Non "ideal" case

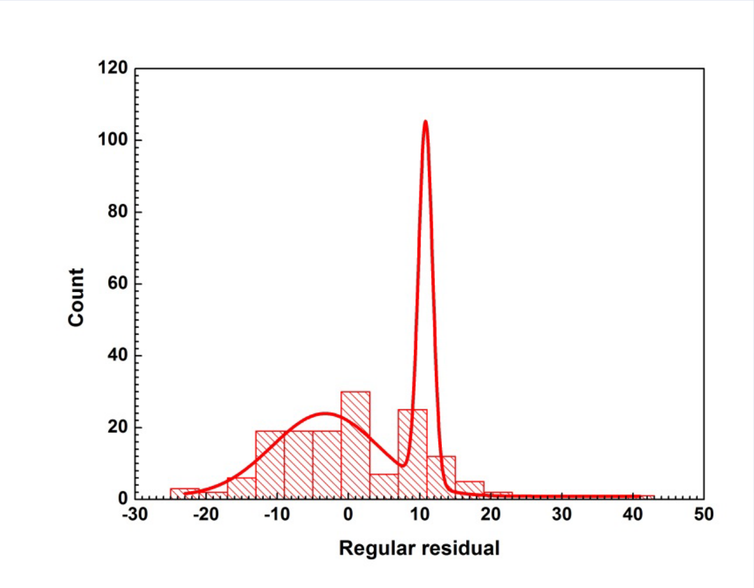


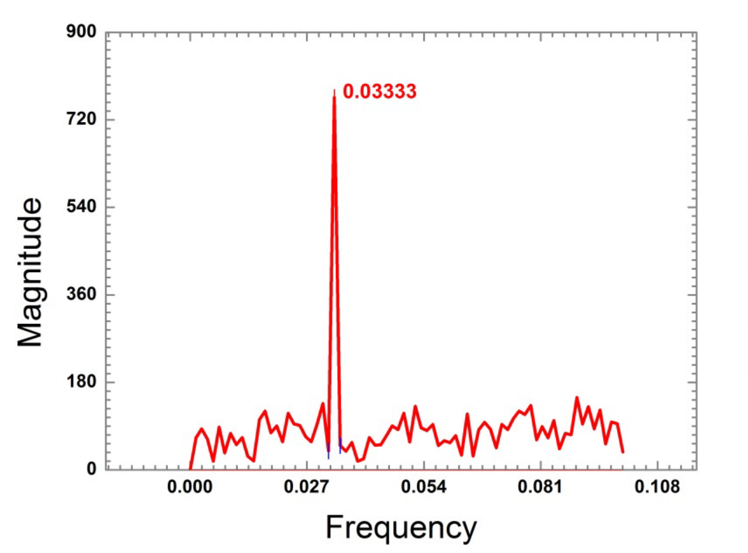


	Clear experiment	Data + "noise"
t_1(s)	177.76	145.89
t_2(s)	30.32	27.94

Fitting. Analysis of the residuals. Non "ideal" case

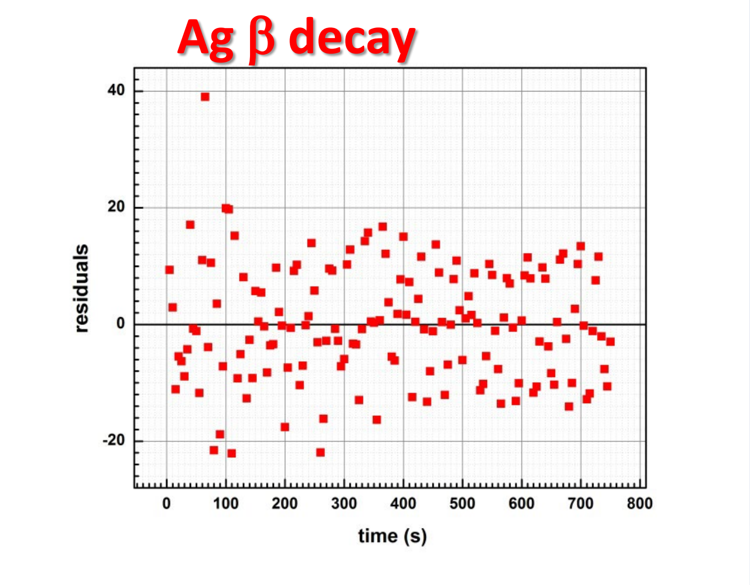


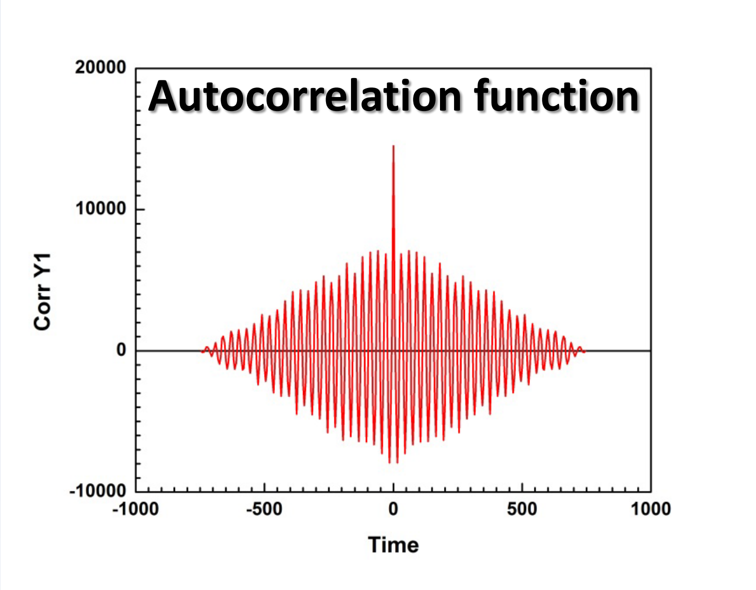




Histogram does not follow the normal distribution and there is frequency of 0.333 is present in spectrum

Fitting. Analysis of the residuals. Non “ideal” case

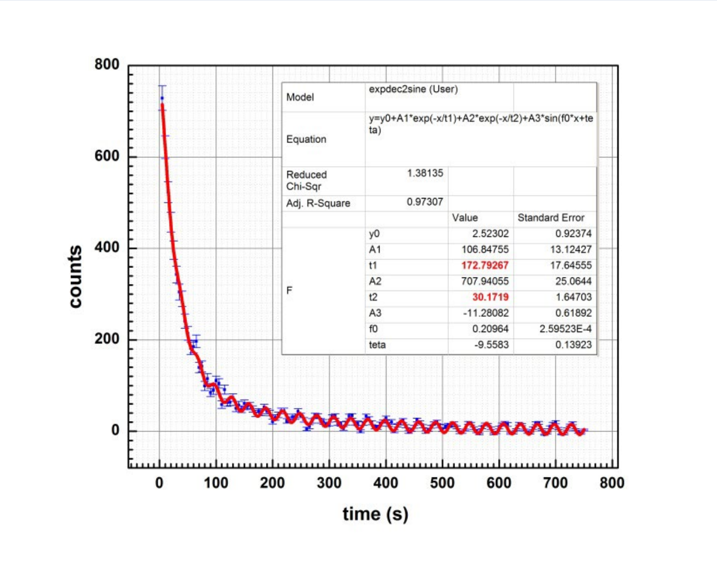


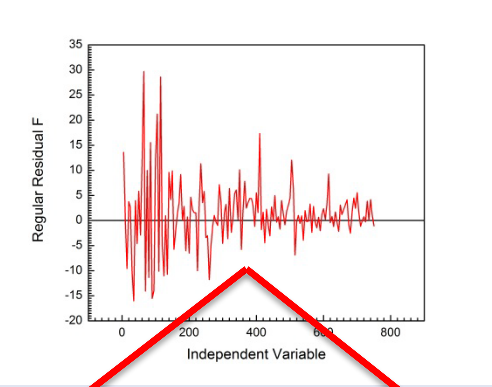


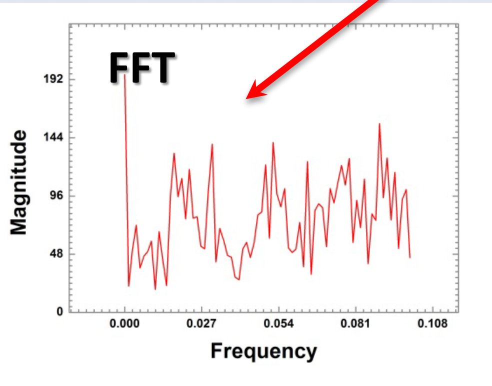
Conclusion: fitting function should be modified by adding an additional term:

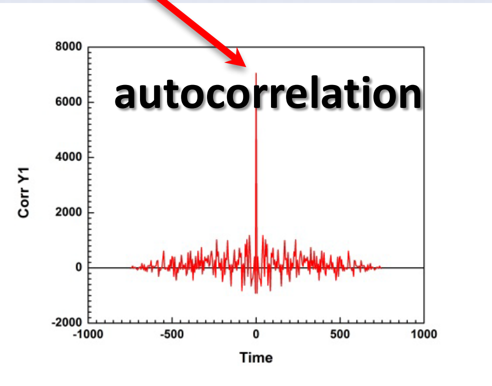
$$y(t) = y_0 + A_1 \exp\left(\frac{-t}{t_1}\right) + A_2 \exp\left(\frac{-t}{t_2}\right) + A_3 \sin(\omega t + \theta)$$

Fitting. Analysis of the residuals. Non "ideal" case







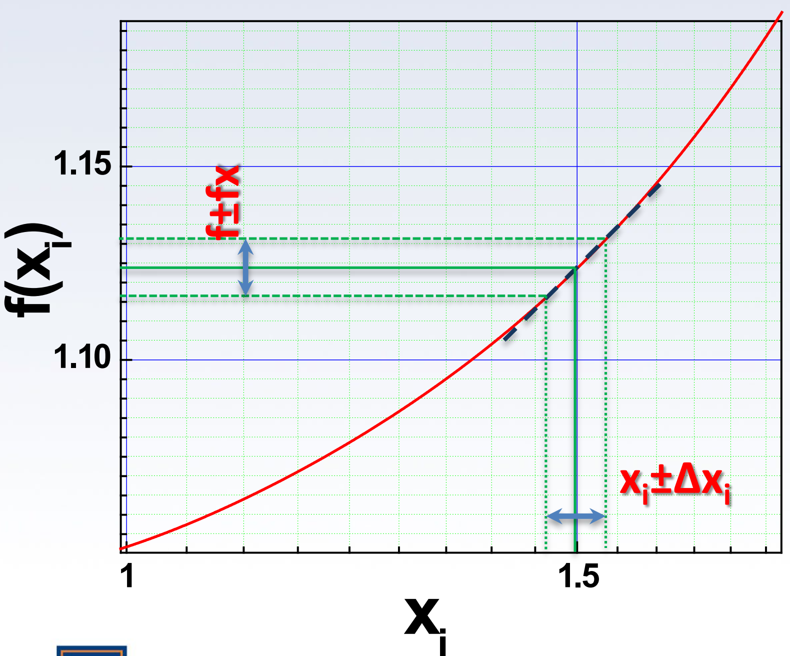


	Clear experiment	Data + noise	Modified fitting
t_1(s)	177.76	145.89	172.79
t_2(s)	30.32	27.94	30.17

Error propagation

y = f(x1, x2 ... xn)

$$\Delta f(x_i, \Delta x_i) = \sqrt{\sum_{i=1}^{n} \left[\frac{\partial f}{\partial x_i}\right]^2 \cdot \Delta x_i^2}$$



Error propagation. Example.

Derive resonance frequency *f* from measured inductance *L±ΔL* and capacitance *C±ΔC*

$$f(L,C)=\frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

$$L_1 = 10\pm 1\text{mH},\quad C_1 = 10\pm 2\mu\text{F}$$

$$\Delta f(L,C,\Delta L,\Delta C)=\sqrt{\left[\frac{\partial f}{\partial L}\right]^2\cdot\Delta L^2+\left[\frac{\partial f}{\partial C}\right]^2\Delta C^2}$$

$$\frac{\partial f}{\partial L}=\frac{-1}{4\pi}C^{-\frac{1}{2}}L^{-\frac{3}{2}};$$

$$\frac{\partial f}{\partial C}=\frac{-1}{4\pi}L^{-\frac{1}{2}}C^{-\frac{3}{2}}$$

Results:

$f(L_1,C_1)$=503.29212104487Hz

Δf=56.26977Hz

$f(L_1,C_1)$=503±56Hz

Presentation of the results. Reports/publications. Condenced Matter Physics.

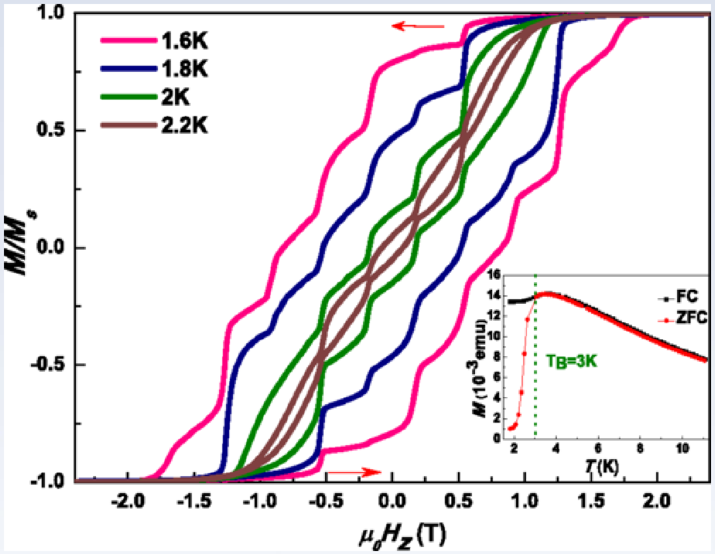


Figure 3.Magnetization (M/Ms) of Mn3 single crystal versus applied magnetic field with the sweeping rate of 0.003 T/s at different temperatures. The inset shows ZFC and FC curves.

Phys. Rev. B 89, 184401

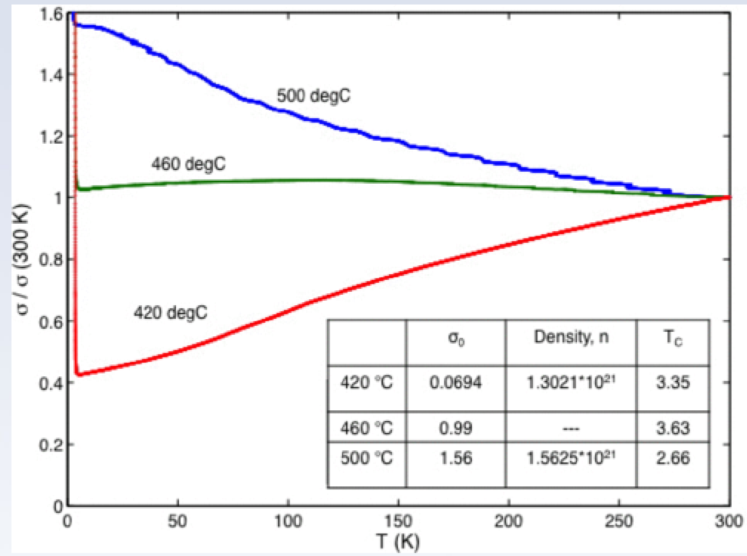


Figure 2. Normalized conductivity vs temperature for three 250-nm-thick K0.33WO3−y films on YSZ substrates. The films are annealed in vacuum at different temperatures, with properties shown in the inset table. The units of T_{anneal} are degrees Celcius, σ0 is given in 1/mΩcm, n in /cm3, and Tc in degrees Kelvin.

Phys. Rev. B 89, 184501

Presentation of the results. Reports/publications. Particle Physics.

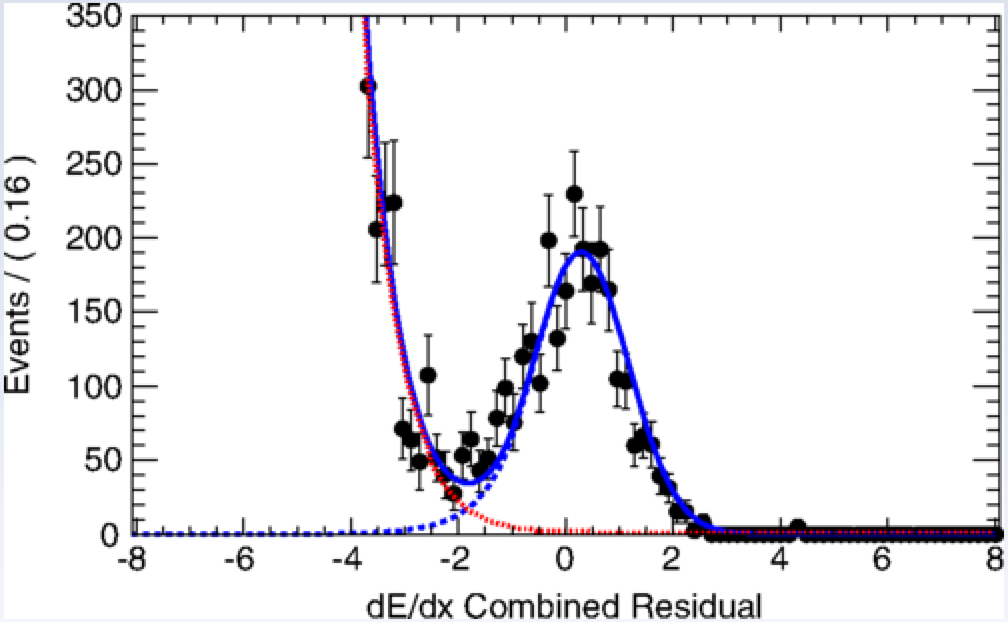


Figure 1. Normalized residuals of the combined dE/dx for antideuteron candidates in the Onpeak ϒ(2S) data sample, with fit PDFs superimposed. Entries have been weighted, as detailed in the text. The solid (blue) line is the total fit, the dashed (blue) line is the d^- signal peak, and the dotted (red) line is the background.

Phys. Rev. D 89, 111102(R)

Presentation of the results. Student Reports.

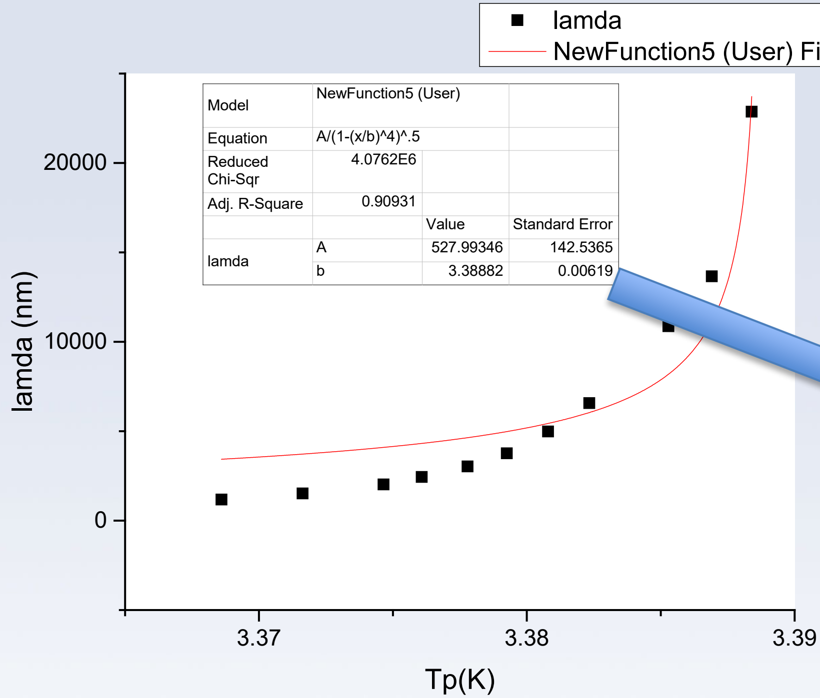
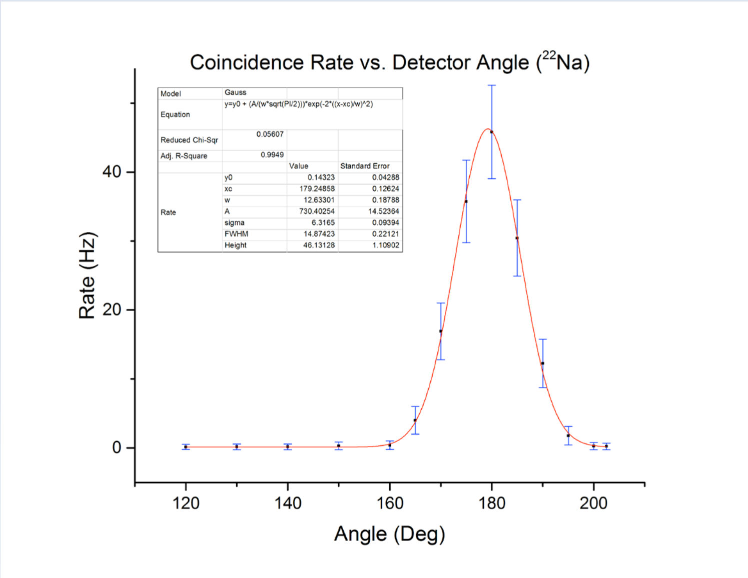


Figure 10(ii): lambda versus T for indium film with thickness 300 nm. Input voltage is 0.2v. Critical temperature(b) and penetration depth(A) at temperature 0 K is determined

Model	NewFunction5 (User)		
Equation	A/(1-(x/b)^4)^.5		
Reduced Chi-Sqr		4.0762E6	
Adj. R-Square		0.90931	
		Value	Standard Error
lamda	A	527.99346	142.5365
lamda	b	3.38882	0.00619

Spring 2014.

Presentation of the results. Student Reports.



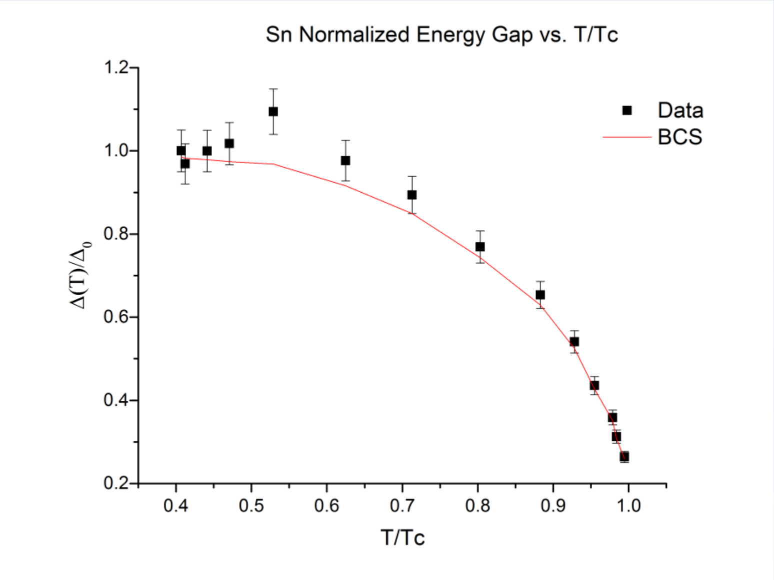


Figure 8: Coincidence Rate vs. Detector Angle for 22Na correlation measurement.

Figure 11: Temperature dependence of energy gap in Sn. Red line is BCS theory

Summer 2019.