

1) **Counting indices:** Show that in d dimensions:

- i) the dimension of the space of skew-symmetric covariant tensors with p indices is $d!/p!(d-p)!$;
- ii) the dimension of the space of symmetric covariant tensors with p indices is $(d+p-1)!/p!(d-1)!$.

2) **Quantum Entanglement:** Two quantum mechanical systems have Hilbert spaces $H^{(1)}$ with basis $\mathbf{e}_1^{(1)}, \dots, \mathbf{e}_m^{(1)}$ and $H^{(2)}$ with basis $\mathbf{e}_1^{(2)}, \dots, \mathbf{e}_n^{(2)}$. The Hilbert space for the combined system is then $H^{(1)} \otimes H^{(2)}$ with basis $\mathbf{e}_i^{(1)} \otimes \mathbf{e}_j^{(2)}$, so the quantum state of the combined system is described by a state

$$\mathbf{a} = a^{ij} \mathbf{e}_i^{(1)} \otimes \mathbf{e}_j^{(2)} \in H^{(1)} \otimes H^{(2)}.$$

If we can find vectors

$$\mathbf{x} = x^i \mathbf{e}_i^{(1)} \in H^{(1)}$$

$$\mathbf{y} = y^j \mathbf{e}_j^{(2)} \in H^{(2)}$$

such that

$$\mathbf{a} = \mathbf{x} \otimes \mathbf{y} = x^i y^j \mathbf{e}_i^{(1)} \otimes \mathbf{e}_j^{(2)}$$

then the tensor \mathbf{a} is said to be *decomposable* and the two quantum systems are said to be *unentangled*. If there are no such vectors the two systems are *entangled* in the sense of the Einstein-Podolski-Rosen (EPR) paradox.

- i) By counting the number of components that are at our disposal \mathbf{a} and in $\mathbf{x} \otimes \mathbf{y}$ find out how many relations the coefficients a_{ij} must satisfy if the state is to be decomposable.
- ii) If the state is decomposable, show that

$$0 = \begin{vmatrix} a^{ij} & a^{il} \\ a^{kj} & a^{kl} \end{vmatrix}$$

for all sets of indices i, j, k, l .

- iii) Using your result from part i) as a reality check, find a subset of the relations from part ii), that constitute a necessary and sufficient set of conditions for the state \mathbf{a} to be decomposable. Include a proof that your set is indeed sufficient.

Since quantum states are really in one-to-one correspondence with *rays* in the Hilbert space, rather than vectors, the set of decomposable states should be thought of as a subset of the complex projective space \mathbf{CP}^{nm-1} , and since it is defined by a finite number of polynomial equations it forms what algebraic geometers call a *variety*. This particular subset is known as the *Segre variety*.

3) **Symmetric integration:** Show that the n -dimensional integral

$$I_{\alpha\beta\gamma\delta} = \int \frac{d^n k}{(2\pi)^n} (k_\alpha k_\beta k_\gamma k_\delta) f(k^2),$$

is equal to

$$A(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})$$

where

$$A = \frac{1}{n(n+2)} \int \frac{d^n k}{(2\pi)^n} (k^2)^2 f(k^2).$$

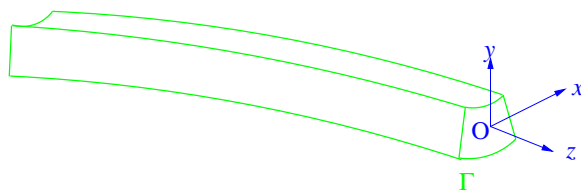
Similarly evaluate

$$I_{\alpha\beta\gamma\delta\epsilon} = \int \frac{d^n k}{(2\pi)^n} (k_\alpha k_\beta k_\gamma k_\delta k_\epsilon) f(k^2).$$

4) **Leonardo da Vinci's problem II:** A steel beam is forged so that its cross section has the shape of a region $\Gamma \in \mathcal{R}^2$. The centroid, O, of each cross section is defined so that

$$\int_{\Gamma} x \, dx \, dy = \int_{\Gamma} y \, dx \, dy = 0,$$

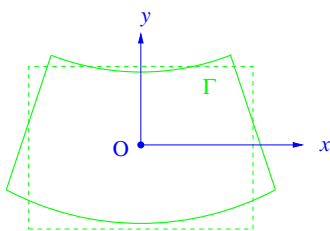
where the co-ordinates x, y are defined with the centroid O as the origin. The beam is slightly bent in the $y - z$ plane so that near a particular cross-section the line of centroids has radius of curvature R . (In the figure this cross section is depicted at the end of the beam. It is actually an interior slice)



Bent beam.

Assume that near O the deformation is such that

$$\begin{aligned} \eta_x &= -\frac{\sigma}{R}xy \\ \eta_y &= \frac{1}{2R} \{ \sigma(x^2 - y^2) - z^2 \} \\ \eta_z &= \frac{1}{R}yz \end{aligned}$$



The deformed cross-section (greatly exaggerated).

Verify that this distortion field does correspond to the beam being bent downwards, and with the line of centroids having radius of curvature R . Notice how, for positive Poisson ratio, the cross section is deformed *anticlastically* — the sides bend *up* as the beam bends *down*. Show that

$$e_{xx} = -\frac{\sigma}{R}y, \quad e_{yy} = -\frac{\sigma}{R}y, \quad e_{zz} = \frac{1}{R}y.$$

Since steel is isotropic, the stresses are derived from the strains *via* the Lamé constants. Show that $\sigma_{zz} = Yy/R$, where Y is Young's modulus, and that all other components of the stress tensor vanish. Deduce from this that the assumed deformation satisfies the force-free surface boundary condition, and so is indeed the way the beam deforms. The total elastic energy is given by

$$E = \iiint_{\text{beam}} \frac{1}{2} e_{ij} c_{ijkl} e_{kl} d^3x.$$

Show that for our bent beam, this reduces to

$$E = \int \frac{YI}{2} \left(\frac{1}{R^2} \right) ds \approx \int \frac{YI}{2} (y'')^2 dz.$$

Here s is the arc-length taken along the line of centroids of the beam,

$$I = \int_{\Gamma} y^2 dx dy$$

is the moment of inertia of the region Γ about an axis through the centroid, and perpendicular both to the length of the beam and the plane into which it is bent. The right-hand-side formula is the expression used many times in 508. Here y denotes the deflection of the beam away from the z axis, and the primes denote differentiation with respect to z or s .

5) Maxwell Stress: Let

$$\Pi_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} |E|^2 \right) + \mu_0 \left(H_i H_j - \frac{1}{2} \delta_{ij} |H|^2 \right).$$

Show that Maxwell's equations lead to

$$(\rho \mathbf{E} + \mathbf{j} \times \mathbf{B})_i + \frac{\partial}{\partial t} \left\{ \frac{1}{c^2} (\mathbf{E} \times \mathbf{H})_i \right\} = \partial_j \Pi_{ji}.$$