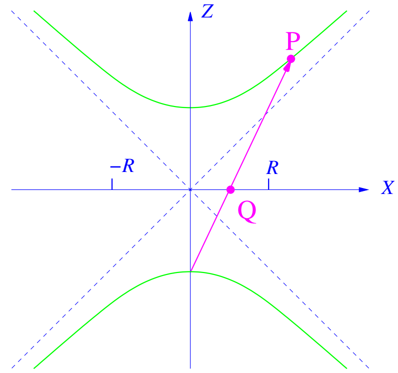
1) Lobachevski space: The hyperbolic plane of Lobachevski geometry can be realized by embedding the $Z \geq R$ branch of the two-sheeted hyperboloid $Z^2 - X^2 - Y^2 = R^2$ into a Minkowski space with metric $ds^2 = -dZ^2 + dX^2 + dY^2$.

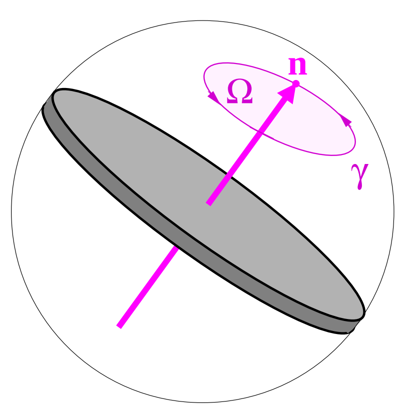


We can parametrize this surface by making an "imaginary radius" version of the stereographic map, in which the point P on the hyperboloid is mapped to the point Q on the X-Y plane. Show that the resulting metric is that of the Poincaré disc model:

$$g(\ ,\ ) = \frac{4R^4}{(R^2 - X^2 - Y^2)^2}(dX \otimes dX + dY \otimes dY), \quad X^2 + Y^2 < R^2$$

2) Flywheel and Rolling Ball: These two problems make use of the area 2-form on the sphere.

a) A flywheel of moment of inertia I can rotate without friction about an axle whose direction is specified by a unit vector $\mathbf{n}$. The flywheel and axle are initially stationary. The direction $\mathbf{n}$ of the axle is made to describe a simple closed curve $\gamma = \partial\Omega$ on the unit sphere, and is then left stationary.



Show that once the axle has returned to rest in its initial direction, the flywheel has also returned to rest, but has rotated through an angle $\theta = \text{Area}(\Omega)$ when compared with its initial orientation. The area of Ω is to be counted as positive if the path γ surrounds it in a clockwise sense, and negative otherwise. Observe that the path γ bounds two regions with opposite orientations. Taking into account that we cannot define the rotation angle at intermediate steps, show that the area of either region can be used to compute θ, the results being physically indistinguishable. (Hint: Use Euler angles and show that the component $L_Z = I(\dot{\psi} + \dot{\phi}\cos\theta)$ of the flywheel's angular momentum along the axle is a constant of the motion.)

b) A ball of unit radius rolls without slipping on a table. The ball moves in such a way that the point in contact with table describes a closed path $\gamma = \partial\Omega$ on the *ball*. (The corresponding path on the *table* will not necessarily be closed.) Show that the final orientation of the ball will be such that it has rotated, when compared with its initial orientation, through an angle $\phi = \text{Area}(\Omega)$ about a vertical axis through its center, As in the previous part, the area is counted positive if γ encircles Ω in an anti-clockwise sense. (Hint: recall the no-slip rolling condition $\dot{\phi} + \dot{\psi}\cos\theta = 0$.)

3) Hopf Invariant: The next pair of exercises explores some physics appearances of the continuum Hopf linking number.

a) The equations governing the motion of an incompressible inviscid fluid are $\nabla \cdot \mathbf{v} = 0$ and Euler's equation

$$\frac{D\mathbf{v}}{Dt} \equiv \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P.$$

Recall that the operator $\partial/\partial t + \mathbf{v} \cdot \nabla$, here written as D/Dt, is called the *convective derivative*.

i) Take the curl of Euler's equation to show that if $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is the *vorticity* then

$$\frac{D\boldsymbol{\omega}}{Dt} \equiv \frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{v}.$$

ii) Combine Euler's equation with part a) to show that

$$\frac{D}{Dt}(\mathbf{v} \cdot \boldsymbol{\omega}) = \nabla \cdot \left\{ \boldsymbol{\omega} \left(\frac{1}{2}\mathbf{v}^2 - P \right) \right\}.$$

iii) Show that if Ω is a volume co-moving with the fluid and f is a scalar function, then

$$\frac{d}{dt} \int_\Omega f(\mathbf{r}, t)\, dV = \int_\Omega \frac{Df}{Dt}\, dV$$

iv) Conclude that when $\boldsymbol{\omega}$ is zero at infinity the *helicity*

$$H = \int \mathbf{v} \cdot \boldsymbol{\omega}\, dV$$

is a constant of the motion.

The helicity measures the Hopf linking number of the vortex lines. The discovery of its conservation by Kieth Moffatt founded the field of *topological fluid dynamics*.

b) Let $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\partial \mathbf{A}/\partial t - \nabla \phi$ be the electric and magnetic field in an incompressible and perfectly conducting fluid. In such a fluid the co-moving electromotive force $\mathbf{E} + \mathbf{v} \times \mathbf{B}$ must vanish everywhere.

i) Use Maxwell's equations to show that

$$\begin{aligned} \frac{\partial \mathbf{A}}{\partial t} &= \mathbf{v} \times (\nabla \times \mathbf{A}) - \nabla \phi, \\ \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B}). \end{aligned}$$

ii) From part a) show that the convective derivative of $\mathbf{A} \cdot \mathbf{B}$ is given by

$$\frac{D}{Dt} (\mathbf{A} \cdot \mathbf{B}) = \nabla \cdot \{\mathbf{B} (\mathbf{A} \cdot \mathbf{v} - \phi)\}.$$

iii) By using the same reasoning as the previous problem, conclude that *Woltjer's invariant*

$$W = \int (\mathbf{A} \cdot \mathbf{B})\, dV = \int \epsilon_{ijk} A_i \partial_j A_k d^3x = \int AF$$

is a constant of the motion, provided that $\mathbf{B}$ is zero at infinity.

This result shows that the Hopf linking number of the magnetic field lines is independent of time. It is an essential ingredient in the geodynamo theory of the Earth's magnetic field.

4) Faraday's Law: Faraday's "flux rule" for computing the electromotive force $\mathcal{E}$ in a circuit containing a thin moving wire is usually derived by the following manipulations:

$$\begin{aligned} \mathcal{E} &\equiv \oint_{\partial\Omega} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r} \\ &= \int_{\Omega} \operatorname{curl} \mathbf{E} \cdot d\mathbf{S} - \oint_{\partial\Omega} \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{r}) \\ &= -\int_{\Omega} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} - \oint_{\partial\Omega} \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{r}) \\ &= -\frac{d}{dt} \int_{\Omega} \mathbf{B} \cdot d\mathbf{S}. \end{aligned}$$

a) Show that if we parameterize the surface Ω as $x^\mu(u, v, \tau)$, with u, v labelling points on Ω, and τ parametrizing the evolution of Ω, then the corresponding manipulations in the covariant differential-form version of Maxwell's equations lead to

$$\frac{d}{d\tau} \int_{\Omega} F = \int_{\Omega} \mathcal{L}_V F = \int_{\partial\Omega} i_V F = -\int_{\partial\Omega} f,$$

where $V^\mu = \partial x^\mu / \partial \tau$ and $f = -i_V F$.

b) Show that if we take τ to be the proper time along the world-line of each element of Ω, then V is the 4-velocity

$$V^\mu = \frac{1}{\sqrt{1-\mathbf{v}^2}}(1, \mathbf{v}),$$

and $f = -i_V F$ becomes the one-form corresponding to the Lorentz-force 4-vector.

It is not clear that the terms in this covariant form of Farday's law can be given any physical interpretation outside the low-velocity limit. When parts of $\partial\Omega$ have different velocities, the relation of the integrals to measurements made at fixed co-ordinate time requires thought.[1]

[1]See E. Marx, *Journal of the Franklin Institute,* **300** (1975) 353-364.