

1) Buckyball spectrum.: Consider the symmetry group of the C_{60} buckyball molecule illustrated on page 194 of the notes.

- Starting from the character table of the orientation-preserving icosohedral group Y (table 5.3), and using the fact that the \mathbb{Z}_2 parity inversion $\sigma : \mathbf{r} \rightarrow -\mathbf{r}$ combines with $g \in Y$ so that $D^{J_g}(\sigma g) = D^{J_g}(g)$, whilst $D^{J_u}(\sigma g) = -D^{J_u}(g)$, write down the character table of the extended group $Y_h = Y \times \mathbb{Z}_2$ that acts as a symmetry on the C_{60} molecule. There are now ten conjugacy classes, and the ten representations will be labelled $A_g, A_u, \text{etc.}$ Verify that your character table has the expected row-orthogonality properties.
- By counting the number of atoms left fixed by each group operation, compute the compound character of the action of Y_h on the C_{60} molecule. (Hint: Examine the pattern of panels on a regulation soccer ball, and deduce that four carbon atoms are left unmoved by operations in the class σC_2 .)
- Use your compound character from part b), to show that the 60-dimensional Hilbert space decomposes as

$$\mathcal{H}_{C_{60}} = A_g \oplus T_{1g} \oplus 2T_{1u} \oplus T_{2g} \oplus 2T_{2u} \oplus 2G_g \oplus 2G_u \oplus 3H_g \oplus 2H_u,$$

consistent with the energy-levels sketched in figure 5.3.

2) Matrix commutators:

- Let $\hat{\lambda}_1$ and $\hat{\lambda}_2$ be hermitian matrices. Show that if we define $\hat{\lambda}_3$ by the relation $[\hat{\lambda}_1, \hat{\lambda}_2] = i\hat{\lambda}_3$, then $\hat{\lambda}_3$ is also a hermitian matrix.
- For the Lie group $O(n)$, the matrices “ $i\hat{\lambda}$ ” are real n -by- n skew symmetric matrices. Show that if A_1 and A_2 are real skew symmetric matrices, then so is $[A_1, A_2]$.
- For the Lie group $Sp(2n, \mathbb{R})$, the $i\hat{\lambda}$ matrices are of the form

$$A = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}$$

where a is a real n -by- n matrix and b and c are symmetric ($a^T = a$ and $b^T = b$) real n -by- n matrices. Show that the commutator of any two matrices of this form is also of this form.

3 Euler angles and SU(2): Parametrize the elements of $SU(2)$ as

$$\begin{aligned} U &= \exp\{-i\phi\hat{\sigma}_3/2\} \exp\{-i\theta\hat{\sigma}_2/2\} \exp\{-i\psi\hat{\sigma}_3/2\}, \\ &= \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix} \begin{pmatrix} \cos \theta/2 & -\sin \theta/2 \\ \sin \theta/2 & \cos \theta/2 \end{pmatrix} \begin{pmatrix} e^{-i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix}, \\ &= \begin{pmatrix} e^{-i(\phi+\psi)/2} \cos \theta/2 & -e^{i(\psi-\phi)/2} \sin \theta/2 \\ e^{i(\phi-\psi)/2} \sin \theta/2 & e^{+i(\psi+\phi)/2} \cos \theta/2 \end{pmatrix}. \end{aligned}$$

- a) Show that Hopf : $S^3 \rightarrow S^2$ is the projection of $S^3 \simeq \text{SU}(2)$ onto the coset space $S^2 \simeq \text{SU}(2)/\text{U}(1)$, where $\text{U}(1)$ is the subgroup $\{\exp(-i\psi\hat{\sigma}_3/2)\}$. Conclude that Hopf takes $(\theta, \phi, \psi) \rightarrow (\theta, \phi)$, where θ and ϕ are spherical polar co-ordinates on the two-sphere.
- b) Show that

$$U^{-1}dU = -\frac{i}{2}\hat{\sigma}_i\Omega_L^i,$$

where

$$\begin{aligned}\Omega_L^1 &= \sin\psi d\theta - \sin\theta \cos\psi d\phi, \\ \Omega_L^2 &= \cos\psi d\theta + \sin\theta \sin\psi d\phi, \\ \Omega_L^3 &= d\psi + \cos\theta d\phi.\end{aligned}$$

Compare these 1-forms with the components

$$\begin{aligned}\omega_X &= \sin\psi \dot{\theta} - \sin\theta \cos\psi \dot{\phi}, \\ \omega_Y &= \cos\psi \dot{\theta} + \sin\theta \sin\psi \dot{\phi}, \\ \omega_Z &= \dot{\psi} + \cos\theta \dot{\phi}.\end{aligned}$$

of the angular velocity $\boldsymbol{\omega}$ of a body with respect to the *body-fixed* XYZ .

- c) (Optional) Now show that

$$dUU^{-1} = -\frac{i}{2}\hat{\sigma}_i\Omega_R^i,$$

where

$$\begin{aligned}\Omega_R^1 &= -\sin\phi d\theta + \sin\theta \cos\psi d\psi, \\ \Omega_R^2 &= \cos\phi d\theta + \sin\theta \sin\psi d\psi, \\ \Omega_R^3 &= d\phi + \cos\theta d\psi,\end{aligned}$$

Compare these 1-forms with components $\omega_x, \omega_y, \omega_z$ of the same angular velocity vector $\boldsymbol{\omega}$, but now with respect to the *space-fixed* xyz frame.

4) Class and group volume:

- a) In the lecture notes I claimed that the volume fraction of the group $\text{SU}(2)$ occupied by rotations through angles lying between θ and $\theta + d\theta$ is $\sin^2(\theta/2)d\theta/\pi$. By considering the geometry of the three-sphere, show that this is correct.
- b) Show that

$$\int_{\text{SU}(2)} \text{tr} [(U^{-1}dU)^3] = 24\pi^2.$$

- c) Suppose we have a map $g : \mathbb{R}^3 \rightarrow \text{SU}(2)$ such that $g(x)$ goes to the identity element at infinity. Consider the integral

$$S[g] = \frac{1}{24\pi^2} \int_{\mathbb{R}^3} \text{tr} [(g^{-1}dg)^3],$$

where the 3-form $\text{tr} (g^{-1}dg)^3$ is the pull-back to \mathbb{R}^3 of the form $\text{tr} [(U^{-1}dU)^3]$ on $\text{SU}(2)$. Show that if we vary $g \rightarrow g + \delta g$, then

$$\delta S[g] = \frac{1}{24\pi^2} \int_{\mathbb{R}^3} d \{3 \text{tr} [(g^{-1}\delta g)(g^{-1}dg)^2]\} = 0,$$

and so $S[g]$ is topological invariant of the map g . Conclude that the functional $S[g]$ is an integer, that integer being the Brouwer degree, or winding number, of the map $g : S^3 \rightarrow S^3$.

5) Campbell-Baker-Hausdorff Formulae: Here are some useful formula for working with exponentials of matrices that do not commute with each other.

- a) Let X and Y be matrices. Show that

$$e^{tX} Y e^{-tX} = Y + t[X, Y] + \frac{1}{2}t^2[X, [X, Y]] + \dots,$$

the terms on the right being the series expansion of $\exp[\text{ad}(tX)]Y$. A proof is sketched in a footnote in the lecture notes, but I want you to fill in the details.

- b) Let X and δX be matrices. Show that

$$\begin{aligned} e^{-X} e^{X+\delta X} &= 1 + \int_0^1 e^{-tX} \delta X e^{tX} dt + O[(\delta X)^2] \\ &= 1 + \delta X - \frac{1}{2}[X, \delta X] + \frac{1}{3!}[X, [X, \delta X]] + \dots \\ &= 1 + \left(\frac{1 - e^{-\text{ad}(X)}}{\text{ad}(X)} \right) \delta X + O[(\delta X)^2] \end{aligned}$$

- c) By expanding out the exponentials, show that

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\text{higher}},$$

where “higher” means terms higher order in X, Y . The next two terms are, in fact, $\frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]]$.

6) SU(3): Here are some simple results that come from playing with the Gell-Mann lambda matrices, as well as practice at decomposing tensor products.

The totally antisymmetric structure constants, f_{ijk} , and a set of totally symmetric constants d_{ijk} are defined by

$$f_{ijk} = \frac{1}{2} \text{tr} (\lambda_i [\lambda_j, \lambda_k]), \quad d_{ijk} = \frac{1}{2} \text{tr} (\lambda_i \{\lambda_j, \lambda_k\}).$$

Let $D_{ij}^8(g)$ be the matrices representing SU(3) in “8” — the eight-dimensional adjoint representation.

a) Show that

$$\begin{aligned} f_{ijk} &= D_{il}^8(g)D_{jm}^8(g)D_{kn}^8(g)f_{lmn}, \\ d_{ijk} &= D_{il}^8(g)D_{jm}^8(g)D_{kn}^8(g)d_{lmn}, \end{aligned} \tag{1}$$

and so f_{ijk} and d_{ijk} are *invariant tensors* in the same sense that δ_{ij} and $\epsilon_{i_1\dots i_n}$ are invariant tensors for SO(n).

b) Let $w_i = f_{ijk}u_jv_k$. Show that if $u_i \rightarrow D_{ij}^8(g)u_j$ and $v_i \rightarrow D_{ij}^8(g)v_j$, then $w_i \rightarrow D_{ij}^8(g)w_j$. Similarly for $w_i = d_{ijk}u_jv_k$. (Hint: show first that the D^8 matrices are real and orthogonal.) Deduce that f_{ijk} and d_{ijk} are *Clebsch-Gordan coefficients* for the $8 \oplus 8$ part of the decomposition

$$8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27.$$

a) Similarly show that $\delta_{\alpha\beta}$ and the lambda matrices $(\lambda_i)_{\alpha\beta}$ can be regarded as Clebsch-Gordan coefficients for the decomposition

$$3 \otimes \overline{3} = 1 \oplus 8.$$

d) Use the graphical method, introduced in class, of plotting weights and peeling off irreps to obtain the tensor product decomposition in part b).