**Physics 509 Homework 0** Prof. M. Stone
**Spring 2021** University of Illinois

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## 1 Index Gymnastics and Einstein Convention

Note that no distinction is made between raised and lowered indices in the problem.

(i) See (ii) with $\mathbf{y} \to \mathbf{x}$.

(ii) $\mathbf{x} \cdot \mathbf{y} = (x^\mu \hat{\mathbf{e}}_\mu)(y^\nu \hat{\mathbf{e}}_\nu) = x^\mu y^\nu \underbrace{\hat{\mathbf{e}}_\mu \hat{\mathbf{e}}_\nu}_{=\delta_{\mu\nu}} = x^\mu y_\mu.$

(iii) Note that $\delta_{\mu\nu}\delta_{\nu\rho} = 1$ if and only if $\mu = \rho$. This agrees with $\delta_{\mu\rho}$ for all values of $\mu$ and $\rho$, hence $\delta_{\mu\nu}\delta_{\nu\rho} = \delta_{\mu\rho}$.

(iv) $a_\mu = a_\nu(\hat{\mathbf{e}}_\nu \cdot \hat{\mathbf{e}}_\mu) = a_\nu \delta_{\mu\nu}$.

(v) $\delta_{\mu\mu} = \sum_{i=1}^{3} 1 = 3$.

(a) Writing out both sides of the expression explicitly, one finds on the LHS

$$\begin{aligned}(A^\mu B_\mu)(C^\nu D_\nu) &= (a_1b_1 + a_2b_2 + a_3b_3)(c_1d_1 + c_2d_2 + c_3d_3) \\ &= a_1b_1c_1d_1 + a_1b_1c_2d_2 + a_1b_1c_3d_3 + a_2b_2c_1d_1 + a_2b_2c_2d_2 + a_2b_2c_3d_3 \\ &\qquad + a_3b_3c_1d_1 + a_3b_3c_2d_2 + a_3b_3c_3d_3,\end{aligned}$$

whereas on the RHS,

$$\begin{aligned}&(A^\mu C^\nu)(B_\mu D_\nu) \\ &\quad = a_1c_1b_1d_1 + a_1c_2b_1d_2 + a_1c_3b_1d_3 + a_2c_1b_2d_1 + a_2c_2b_2d_2 + a_2c_3b_2d_3 \\ &\qquad + a_3c_1b_3d_1 + a_3c_2b_3d_2 + a_3c_3b_3d_3.\end{aligned}$$

These expressions are clearly equal.

(b) Suppose $A_{\mu\nu} = -A_{\mu\nu}$ and $B^{\mu\nu} = B^{\nu\mu}$. Then writing out terms explicitly yields

$$A_{\mu\nu}B^{\mu\nu} = a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} + a_{21}b_{21} + a_{22}b_{22} + a_{23}b_{23} + a_{31}b_{31} + a_{32}b_{32} + a_{33}b_{33}.$$

Cancelling all the diagonal terms since $a_{ii} = -a_{ii} \implies a_{ii} = 0$ for any $i$ and replacing $a_{ji} = -a_{ij}$ and $b_{ji} = b_{ij}$ whenever $i < j$ yields

$$A_{\mu\nu} = a_{12}b_{12} + a_{13}b_{13} + a_{23}b_{23} - a_{12}b_{12} - a_{13}b_{13} - a_{23}b_{23} = 0.$$

This can be shown more concisely by relabeling indices:

$$A_{\mu\nu}B^{\mu\nu} \xmapsto{\mu\leftrightarrow\nu} A_{\nu\mu}B^{\nu\mu} = -A_{\mu\nu}B^{\mu\nu} \implies A_{\mu\nu}B^{\mu\nu} = 0.$$

The last equality follows since simply relabelling indices should not change the result.

## 2 Antisymmetry

(a) It suffices to show that 123, 231, and 312 are all even permutations of 123:

$$\begin{aligned}
&123 \xmapsto{\text{id}} 123\\
&123 \xmapsto{(12)} 213 \xmapsto{(23)} 231\\
&123 \xmapsto{(23)} 132 \xmapsto{(12)} 312.
\end{aligned}$$

Each of these contain an even number of transpositions and are therefore even permutations: $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$.

In contrast,

$$1234 \xmapsto{(12)} 2134 \xmapsto{(23)} 2314 \xmapsto{(34)} 2341$$

contains an odd number of transpositions; therefore, $\epsilon_{1234} = -\epsilon_{2341}$.

(b) First, note that only the relative permutations between unprimed and primed indices matters since

$$\epsilon_{ijk}\epsilon_{i'j'k'} = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) = \operatorname{sgn}(\sigma\tau),$$

where $\sigma$ and $\tau$ act on unprimed and primed indices respectively. Without loss of generality let $\tau$ denote a permutation of the primed indices relative to $ijk$. Then

$$\begin{aligned}
\epsilon_{ijk}\epsilon_{i'j'k'} &= \sum_{\tau\in S_3} \operatorname{sgn}(\tau)\delta_{i\tau(i')}\delta_{j\tau(j')}\delta_{k\tau(k')}\\
&= \delta_{ii'}\delta_{jj'}\delta_{kk'} - \delta_{ij'}\delta_{ji'}\delta_{kk'} + \delta_{ij'}\delta_{jk'}\delta_{ki'} - \delta_{ik'}\delta_{jj'}\delta_{ki'}\\
&\quad + \delta_{ik'}\delta_{ji'}\delta_{kj'} - \delta_{ii'}\delta_{jk'}\delta_{kj'}.
\end{aligned}$$

(c) Setting $i = i'$ in part (b) yields

$$\epsilon_{ijk}\epsilon_{ij'k'} = \delta_{jj'}\delta_{kk'} - \delta_{jk'}\delta_{kj'}. \tag{1}$$

(d) The result from (b) clearly generalizes to

$$\epsilon_{ijk\ell}\epsilon_{i'j'k'\ell'} = \sum_{\tau\in S_4} \operatorname{sgn}(\tau)\delta_{i\tau(i')}\delta_{j\tau(j')}\delta_{k\tau(k')}\delta_{\ell\tau(\ell')}.$$

Setting $i = i'$, we recover a similar result to that of part (b) (as expected):

$$\epsilon_{ijk\ell}\epsilon_{i'j'k'\ell'} = \delta_{ii'}\delta_{jj'}\delta_{kk'} - \delta_{ij'}\delta_{ji'}\delta_{kk'} + \delta_{ij'}\delta_{jk'}\delta_{ki'} - \delta_{ik'}\delta_{jj'}\delta_{ki'} + \delta_{ik'}\delta_{ji'}\delta_{kj'} - \delta_{ii'}\delta_{jk'}\delta_{kj'}.$$

## 3 Vector Products

(i) By definition, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_k(\epsilon_{ijk} b_i c_j) = \epsilon_{ijk} a_k b_i c_j$. This is clearly invariant under any even permutation of indices which shows that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

(ii) Plugging in the definition and using our earlier results,

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \mathbf{a} \times (\epsilon_{ijk} b_j c_k \hat{\mathbf{e}}_i) \\ &= \epsilon_{ij'k'}\epsilon_{ijk} a_{k'} b_j c_k \hat{\mathbf{e}}_{j'} \\ &= (\delta_{jj'}\delta_{kk'} - \delta_{jk'}\delta_{kj'}) a_{k'} b_j c_k \hat{\mathbf{e}}_{j'} && \text{(by (1))} \\ &= a_k c_k b_j \hat{\mathbf{e}}_j - a_j b_j c_k \hat{\mathbf{e}}_k. \end{aligned}$$

Expressing this in vector notation yields

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}). \tag{2}$$

(iii) Again, using index notation and previous identities,

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= (\epsilon_{ijk} a_j b_k \hat{\mathbf{e}}_i) \cdot (\epsilon_{i'j'k'} c_{j'} d_{k'} \hat{\mathbf{e}}_{i'}) \\ &= (\epsilon_{ijk} a_j b_k)(\epsilon_{ij'k'} c_{j'} d_{k'}) && (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_{i'} = \delta_{ii'}) \\ &= (\delta_{jj'}\delta_{kk'} - \delta_{jk'}\delta_{kj'}) a_j b_k c_{j'} d_{k'} && \text{(by (1))} \\ &= a_j b_k c_j d_k - a_j b_k c_k d_j. \end{aligned}$$

Expressing this in vector notation yields

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}). \tag{3}$$

(iv) First consider the spherical law of cosines,

$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(C), \tag{4}$$

where the arc lengths (angles between the unit vectors), $a$, $b$, $c$, and angle $C$ are shown in

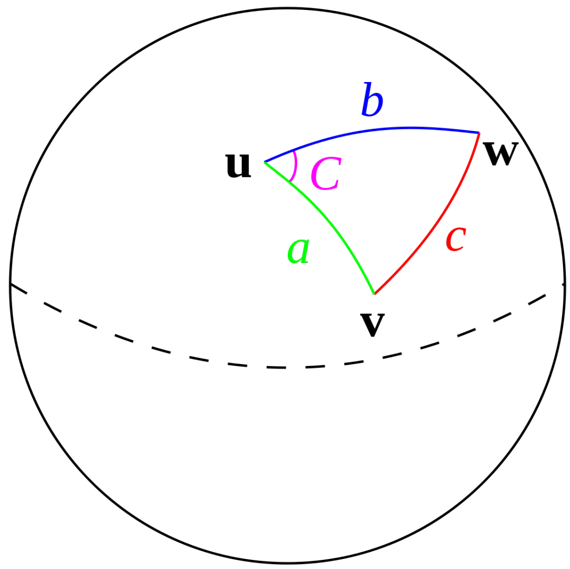


Figure 1: Definitions of arc lengths, angles, and vectors used in the derivation of the law of spherical cosines. Note that the vectors $\mathbf{u}$, $\mathbf{w}$, and $\mathbf{v}$ are placed at the origin (center of the sphere) and are of unit length.

figure 1. Using the familiar identities,

$$\mathbf{a} \cdot \mathbf{b} = ab\cos(\theta) \quad \text{and} \quad |\mathbf{a} \times \mathbf{b}| = ab\sin(\theta),$$

and making the formal substitutions $\mathbf{a} \to \mathbf{u}$, $\mathbf{b} \to \mathbf{v}$, $\mathbf{c} \to \mathbf{u}$, and $\mathbf{d} \to \mathbf{w}$ (so our notation is consistent with that in figure 1), the LHS of equation (3) can be written as

$$(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{w}) = \sin(a)\sin(b)\cos(C).$$

The RHS of equation (3) yields

$$(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{w}) - (\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{w}) = \cos(c) - \cos(a)\cos(b),$$

which rearranges to the desired result (equation (4)).

The spherical law of sines is

$$\frac{\sin(A)}{\sin(a)} = \frac{\sin(B)}{\sin(b)} = \frac{\sin(C)}{\sin(c)}, \tag{5}$$

where $a$, $b$, and $c$ are the arcs on the surface of the sphere (equivalently, the corresponding angles between $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ since the sphere is of unit radius) and $A$, $B$, and $C$ are the spherical angles opposite their respective arcs (e.g., the relationship between $c$ and $C$ is depicted in figure 1).

Following the hint provided, we first prove the identity:

$$\mathbf{a}\cdot[(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c})]=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}). \tag{6}$$

Plugging in the definition of the cross-product,

$$\begin{aligned}
\mathbf{a}\cdot[(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c})] &= \mathbf{a}\cdot\left[(\epsilon_{ijk}a_jb_k\hat{\mathbf{e}}_i)\times(\epsilon_{i'j'k'}a_{j'}b_{k'}\hat{\mathbf{e}}_{i'})\right]\\
&= \mathbf{a}\cdot\left[\epsilon_{\ell ii'}\epsilon_{ijk}\epsilon_{i'j'k'}a_jb_ka_{j'}b_{k'}\hat{\mathbf{e}}_\ell\right]\\
&= \epsilon_{\ell ii'}\epsilon_{ijk}\epsilon_{i'j'k'}a_\ell a_jb_ka_{j'}b_{k'}\\
&= (\delta_{i'j}\delta_{\ell k}-\delta_{i'k}\delta_{\ell j})\epsilon_{i'j'k'}a_\ell a_jb_ka_{j'}b_{k'} && \text{(by (1))}\\
&= \underbrace{\epsilon_{jj'k'}a_ka_ja_{j'}b_kc_{k'}}_{\propto(\mathbf{a}\times\mathbf{a})\cdot\mathbf{c}=0}\underbrace{-\epsilon_{kj'k'}a_\ell a_\ell a_{j'}b_kc_{k'}}_{=\epsilon_{j'kk'}a_{j'}b_kc_{k'}}\\
&= a_i(\epsilon_{ijk}b_jc_k) && \text{(relabelling indices).}
\end{aligned}$$

Note, in the second to last line, $a_\ell a_\ell=1$ since these are unit vectors. This establishes identity (6).

Using this formula with $\mathbf{a}\to\mathbf{u}$, $\mathbf{b}\to\mathbf{v}$, and $\mathbf{c}\to\mathbf{w}$ yields

$$\mathbf{u}\cdot[(\mathbf{u}\times\mathbf{v})\times(\mathbf{u}\times\mathbf{w})]=\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}). \tag{7}$$

The RHS of (7) is invariant under even permutations of vectors (from part (i)). This implies

$$\begin{aligned}
&|(\mathbf{u}\times\mathbf{v})\times(\mathbf{u}\times\mathbf{w})|=|(\mathbf{v}\times\mathbf{w})\times(\mathbf{v}\times\mathbf{u})|=|(\mathbf{w}\times\mathbf{u})\times(\mathbf{w}\times\mathbf{v})|\\
&\qquad\implies \sin(a)\sin(b)\sin(C)=\sin(a)\sin(c)\sin(B)=\sin(b)\sin(c)\sin(A),
\end{aligned}$$

which reduces to the desired result, equation (5). Note that the angle between $(\mathbf{u}\times\mathbf{v})$ and $(\mathbf{u}\times\mathbf{w})$, for example, is just $C$.

## 4 Bernoulli and Vector Products

Let's first rewrite the expression $\mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{v})$ in index notation.

$$\begin{aligned}
\mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{v}) &= \mathbf{u}\times(\epsilon_{ijk}(\partial_iv_j)\hat{\mathbf{e}}_k)\\
&= \epsilon_{i'j'k'}u_{i'}\left(\epsilon_{ijk}(\partial_iv_j)\hat{\mathbf{e}}_k\right)_{j'}\hat{\mathbf{e}}_{k'}\\
&= \epsilon_{i'kk'}\epsilon_{ijk}u_{i'}(\partial_iv_j)\hat{\mathbf{e}}_{k'}\\
&= -\epsilon_{ki'k'}\epsilon_{kij}u_{i'}(\partial_iv_j)\hat{\mathbf{e}}_{k'} && \text{(re-order indices)}\\
&= -(\delta_{ii'}\delta_{jk'}-\delta_{ik'}\delta_{jk'})u_{i'}(\partial_iv_j)\hat{\mathbf{e}}_{k'} && \text{(by (1))}\\
&= u_j(\partial_iv_j)\hat{\mathbf{e}}_i-u_i(\partial_iv_j)\hat{\mathbf{e}}_j.
\end{aligned}$$

This expression is sometimes written using Feynman's subscript notation,

$$\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \nabla_{\mathbf{v}}(\mathbf{u} \cdot \mathbf{v}) - (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{v},$$

where $\nabla_{\mathbf{v}}$ acts only on the $\mathbf{v}$ coordinates to the right. Using $\frac{1}{2}\nabla \mathbf{v}^2 = v_i(\partial_j v_i)\hat{\mathbf{e}}_j$, we can write

$$\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \frac{1}{2}\nabla \mathbf{v}^2 - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}. \tag{8}$$

Using this identity, Euler's equation for fluid motion,

$$\dot{\mathbf{v}} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla} h$$

becomes

$$\dot{\mathbf{v}} - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) + \frac{1}{2}\nabla \mathbf{v}^2 = -\boldsymbol{\nabla} h \implies \dot{\mathbf{v}} - \mathbf{v} \times \boldsymbol{\omega} = -\nabla\left(\frac{1}{2}\mathbf{v}^2 + h\right),$$

where the final expression has been written in terms of the vorticity, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$.

For steady flow ($\dot{\mathbf{v}} = \mathbf{0}$), the quantity $\frac{1}{2}\mathbf{v}^2 + h$ is constant along streamlines since

$$-\mathbf{v} \cdot \nabla\left(\frac{1}{2}\mathbf{v}^2 + h\right) = \mathbf{v} \cdot (\mathbf{v} \times \boldsymbol{\omega}) = 0.$$

## 5 Antisymmetry and Determinants

(a) Given the definition of the determinant,

$$\det(\mathbf{A}) = \epsilon_{j_1 j_2 \ldots j_n} A_{1 j_1} A_{2 j_2} \ldots A_{n j_n}, \tag{9}$$

relabel the indices by a permutation; i.e., by $\sigma$ such that $\sigma(k) = i_k$.

$$\begin{aligned}
\det(\mathbf{A}) &= \epsilon_{j_{\sigma(1)} j_{\sigma(2)} \cdots j_{\sigma(n)}} A_{\sigma(1) j_{\sigma(1)}} A_{\sigma(2) j_{\sigma(2)}} \ldots A_{\sigma(n) j_{\sigma(n)}} \\
&= \epsilon_{j_{i_1} j_{i_2} \cdots j_{i_n}} A_{i_1 j_{i_1}} A_{i_2 j_{i_2}} \ldots A_{i_n j_{i_n}} \\
&= \epsilon_{i_1 i_2 \ldots i_n} \epsilon_{j_1 j_2 \ldots j_n} A_{i_1 j_1} A_{i_2 j_2} \ldots A_{i_n j_n}.
\end{aligned}$$

In the last line, the double subscripts, $j_{i_k}$ terms, have been relabelled to $j_k$ terms. This leaves the product of matrix elements unchanged while introducing a factor of $\epsilon_{i_1 i_2 \ldots i_n}$ from reordering the $\epsilon_{j_{i_1} j_{i_2} \cdots j_{i_n}}$ term. This establishes the desired result,

$$\epsilon_{i_1 i_2 \ldots i_n} \det(\mathbf{A}) = \epsilon_{j_1 j_2 \ldots j_n} A_{i_1 j_1} A_{i_2 j_2} \ldots A_{i_n j_n}. \tag{10}$$

This result can be used to show the Cauchy-Binet formula, $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

$$\begin{aligned}
\det(\mathbf{AB}) &= \epsilon_{j_1 j_2 \dots j_n} A_{1k_1} B_{k_1 j_1} A_{2k_2} B_{k_2 j_2} \dots A_{nk_n} B_{k_n j_n} \\
&= A_{1k_1} A_{2k_2} \dots A_{nk_n} \underbrace{(\epsilon_{j_1 j_2 \dots j_n} B_{k_1 j_1} B_{k_2 j_2} \dots B_{k_n j_n})}_{=\epsilon_{k_1 k_2 \dots k_n} \det(\mathbf{B})} && \text{(by (10))} \\
&= \underbrace{\epsilon_{k_1 k_2 \dots k_n} A_{1k_1} A_{2k_2} \dots A_{nk_n}}_{=\det(\mathbf{A})} \det(\mathbf{B}) && \text{(by (9))} \\
&= \det(\mathbf{A})\det(\mathbf{B}).
\end{aligned}$$

(b) We now repeat the above exercise but using the language of differential forms.

(i) Since $V$ is $n$ dimensional, $\{\omega \,|\, \omega : V^n \to \mathbb{C}\}$ forms a one-dimensional vector space over $\mathbb{C}$. Hence, there is only one form up to multiplicative constant.

(ii) Now we want to show $\{\mathbf{x}_k\}_{k=1}^n$ are linearly independent if and only if $\omega(\mathbf{x}_1, \dots, \mathbf{x}_n) \neq 0$. Or equivalently, $\{\mathbf{x}_k\}_{k=1}^n$ linearly dependent if and only if $\omega(\mathbf{x}_1, \dots, \mathbf{x}_n) = 0$ (this is just the contrapositive). For convenience of notation, write $\mathbf{x}_1$ as $\mathbf{x}^{(1)}$.

( $\Longrightarrow$ ) First, suppose $\omega(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}) = 0$. Define the matrix

$$\mathbf{X} = (\mathbf{x}^{(1)}\ \mathbf{x}^{(2)}\ \dots\ \mathbf{x}^{(n)}).$$

Then,

$$\begin{aligned}
\omega(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}) &= \omega(x_{k_1}^{(1)} \hat{\mathbf{e}}_{k_1}, x_{k_2}^{(2)} \hat{\mathbf{e}}_{k_2}, \dots, x_{k_n}^{(n)} \hat{\mathbf{e}}_{k_n}) \\
&= x_{k_1}^{(1)} x_{k_2}^{(2)} \dots x_{k_n}^{(n)} \omega(\hat{\mathbf{e}}_{k_1}, \hat{\mathbf{e}}_{k_2}, \dots, \hat{\mathbf{e}}_{k_n}) \\
&= \underbrace{\epsilon_{k_1 k_2 \dots k_n} x_{k_1}^{(1)} x_{k_2}^{(2)} \dots x_{k_n}^{(n)}}_{=\det(\mathbf{X})} \underbrace{\omega(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \dots, \hat{\mathbf{e}}_n)}_{=1}.
\end{aligned}$$

This shows that if $\omega(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}) = 0$ then $\det(\mathbf{X}) = 0$ which implies that $\{\mathbf{x}_k\}_{k=1}^n$ are linearly dependent.

( $\Longleftarrow$ ) Conversely, if $\{\mathbf{x}_k\}_{k=1}^n$ are linearly dependent then, without loss of generality, we can write $\mathbf{x}_1 = \sum_{k=2}^n c_k \mathbf{x}_k$ for some coefficients $c_k$. Then

$$\omega(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) = \sum_{k=2}^n c_k \omega(\mathbf{x}_k, \mathbf{x}_2, \dots, \mathbf{x}_n) = 0.$$

Every term in the sum is zero since the (antisymmetric) form contains repeated elements; hence, the sum is identically zero.

Now define the determinant of the linear map $A : V \to V$ by

$$(\det \mathbf{A})\, \omega(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \omega(\mathbf{A}\mathbf{x}_1, \mathbf{A}\mathbf{x}_2, \ldots, \mathbf{A}\mathbf{x}_n). \tag{11}$$

Writing everything in terms of the standard basis, $\mathbf{x}^{(k)} = x_i^{(k)}\hat{\mathbf{e}}_i$, $\mathbf{A}\mathbf{x}^{(k)} = A_{ij}x_j^{(k)}\hat{\mathbf{e}}_i$, and using $\omega(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_n) = 1$, one finds

$$\begin{aligned}
\omega(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}) &= \omega(x_{j_1}^{(1)}\hat{\mathbf{e}}_{j_2}, x_{j_2}^{(2)}\hat{\mathbf{e}}_{j_2}, \ldots, x_{j_n}^{(n)}\hat{\mathbf{e}}_{j_n}) \\
&= x_{j_1}^{(1)} x_{j_2}^{(2)} \ldots x_{j_n}^{(n)} \omega(\hat{\mathbf{e}}_{j_2}, \hat{\mathbf{e}}_{j_2}, \ldots, \hat{\mathbf{e}}_{j_n}) && (\omega \text{ is multilinear}) \\
&= \epsilon_{j_1 j_2 \ldots j_n} x_{j_1}^{(1)} x_{j_2}^{(2)} \ldots x_{j_n}^{(n)} \underbrace{\omega(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_n)}_{=1} && (\text{by skew-symmetry}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\omega(\mathbf{A}\mathbf{x}_1, \mathbf{A}\mathbf{x}_2, \ldots, \mathbf{A}\mathbf{x}_n) &= \omega(A_{i_1 j_1} x_{j_1}^{(1)} \hat{\mathbf{e}}_{i_1}, A_{i_2 j_2} x_{j_2}^{(2)} \hat{\mathbf{e}}_{i_2}, \ldots, A_{i_n j_n} x_{j_n}^{(n)} \hat{\mathbf{e}}_{i_n}) \\
&= A_{i_1 j_1} x_{j_1}^{(1)} A_{i_2 j_2} x_{j_2}^{(2)} \ldots A_{i_n j_n} x_{j_n}^{(n)} \omega(\hat{\mathbf{e}}_{i_1}, \hat{\mathbf{e}}_{i_2}, \ldots, \hat{\mathbf{e}}_{i_n}) \\
&= \epsilon_{i_1 i_2 \ldots i_n} A_{i_1 j_1} A_{i_2 j_2} \ldots A_{i_n j_n} x_{j_1}^{(1)} x_{j_2}^{(2)} \ldots x_{j_n}^{(n)} \underbrace{\omega(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_n)}_{=1}.
\end{aligned}$$

Using these expressions in equation (11) reduces to

$$\det(\mathbf{A}) = \epsilon_{i_1 i_2 \ldots i_n} \epsilon_{j_1 j_2 \ldots j_n} A_{i_1 j_1} A_{i_2 j_2} \ldots A_{i_n j_n},$$

which agrees with equation (10).

The proof of the Cauchy-Binet formula is now trivial:

$$\begin{aligned}
(\det \mathbf{A})(\det \mathbf{B})\omega(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) &= (\det \mathbf{A})\omega(\mathbf{B}\mathbf{x}_1, \mathbf{B}\mathbf{x}_2, \ldots, \mathbf{B}\mathbf{x}_n) \\
&= \omega(\mathbf{A}\mathbf{B}\mathbf{x}_1, \mathbf{A}\mathbf{B}\mathbf{x}_2, \ldots, \mathbf{A}\mathbf{B}\mathbf{x}_n) \\
&= (\det \mathbf{A}\mathbf{B})\omega(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n).
\end{aligned}$$