Physics 509 Homework 4 Professor M. Stone

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1 Old Exam Problem

(a) Taking the exterior derivative of ω,

$$d\omega = d\left(\frac{1}{r^3}\right)\wedge(xdy\wedge dz + zdx\wedge dy + ydz\wedge dx) + \left(\frac{1}{r^3}\right)\wedge d\left(xdy\wedge dz + zdx\wedge dy + ydz\wedge dx\right).$$

But

$$\begin{aligned} d\left(\frac{1}{r^3}\right) &= d\left(x^2+y^2+z^2\right)^{-3/2} \\ &= -3\left(x^2+y^2+z^2\right)^{-5/2}(xdx+ydy+zdz) \qquad (df = (\partial_i f)dx^i) \\ &= -3\left(\frac{1}{r^5}\right)(xdx+ydy+zdz). \end{aligned}$$

Therefore, we find

$$d\omega = -\frac{3}{r^5}\underbrace{\left(x^2+y^2+z^2\right)}_{=r^2}dx\wedge dy\wedge dz + \frac{3}{r^3}dx\wedge dy\wedge dz = 0,$$

where we have used the fact that terms like $dx\wedge dx$ vanish because of the antisymmetry of the wedge product. Hence, ω is closed.

(b) Using a simple change of variables, one finds

$$\Phi = \int_P \omega = \int_{\mathbb{R}^2}\frac{dx\wedge dy}{(1+x^2+y^2)^{3/2}} = \int_0^{2\pi}\int_0^\infty \frac{r\,drd\theta}{(1+r^2)^{3/2}} = 2\pi\underbrace{\int_0^\infty\frac{r\,dr}{(1+r^2)^{3/2}}}_{=1}.$$

Hence $\Phi = 2\pi$.

(c) The most direct method to solve this problem is to just plug into the general formula for the pullback of a p-form (see equation (3)). Using the formula and evaluating all the partial derivatives, one finds $\varphi^*\omega = \sin\theta\, d\theta\wedge d\phi$. However, there are other ways one might go about solving this problem. I've included a few below.[1]

Recall that the pullback of a form acting on a vector field is just the form evaluated on the

[1] These are really all the same.

pushed forward vector field; i.e.,[2]

$$(\varphi^*\omega)(X) = \omega(\varphi_* X), \tag{1}$$

where the pushforward is defined as $\varphi_* : T_x M \to T_{\varphi(x)} N$ such that

$$(\varphi_* X)^\mu = \frac{\partial \xi^\mu}{\partial x^\nu} X^\nu. \tag{2}$$

In our case, we need to push forward a vector from the sphere S^2 to one in $\mathbb{R}^3$ that can be acted upon by the 2-form ω. In addition to directly plugging into the definition, here are two possible ways to perform this calculation. The first method yields the evaluation of the form pulled back to the new manifold whereas the second gives the pulled back form itself.

Method 1: Using equations (1) and (4), one finds the evaluation of the 2-form on S^2 to be[3]

$$\begin{aligned}
(\varphi^*\omega)(X) &= \omega(\varphi_* X) \\
&= \left[\frac{1}{R^3}(xdydz + ydzdx + zdxdy)\right]\left(X^\theta \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + X^\theta \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} + X^\theta \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z},\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. X^\phi \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + X^\phi \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} + X^\phi \frac{\partial z}{\partial \phi}\frac{\partial}{\partial z}\right) \\
&= \left[x\left(\frac{\partial y}{\partial \theta}\frac{\partial z}{\partial \phi} - \frac{\partial z}{\partial \theta}\frac{\partial y}{\partial \phi}\right) + y\left(\frac{\partial z}{\partial \theta}\frac{\partial x}{\partial \phi} - \frac{\partial x}{\partial \theta}\frac{\partial z}{\partial \phi}\right) + z\left(\frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \phi}\right)\right]\frac{X^\theta X^\phi}{R^3}.
\end{aligned}$$

After plugging in for x, y, and z, evaluating all the partial derivatives, and simplifying, one finds that

$$(\varphi^*\omega)(X) = \sin\theta\, X^\theta X^\phi \implies \varphi^*\omega = \sin\theta\, d\theta \wedge d\phi.$$

This method just naïvely plugged in the definitions from chapter 12 of the textbook.

Method 2: Given the change of variables

$$\begin{aligned}
x &= R\cos\phi\sin\theta \\
y &= R\sin\phi\sin\theta \\
z &= R\cos\theta,
\end{aligned}$$

we can apply the chain rule to write

$$\begin{aligned}
dx &= \left(\frac{\partial x}{\partial \theta}\right) d\theta + \left(\frac{\partial x}{\partial \phi}\right) d\phi = (-R\sin\phi\sin\theta)\, d\phi + (\cos\phi\cos\theta)\, d\theta \\
dy &= \left(\frac{\partial y}{\partial \theta}\right) d\theta + \left(\frac{\partial y}{\partial \phi}\right) d\phi = (R\cos\phi\sin\theta)\, d\phi + (R\sin\phi\cos\theta)\, d\theta
\end{aligned}$$

[2] See equation (12.30) in the textbook.
[3] As usual, I will stop writing the wedge product explicitly.

$$dz = \left(\frac{\partial z}{\partial \theta}\right) d\theta + \left(\frac{\partial z}{\partial \phi}\right) d\phi = (-R\sin\theta)\, d\theta.$$

Plugging these into ω and simplifying one finds

$$\begin{aligned}\varphi_*\omega &= \left[x\left(\frac{\partial y}{\partial \theta}\frac{\partial z}{\partial \phi} - \frac{\partial z}{\partial \theta}\frac{\partial y}{\partial \phi}\right) + y\left(\frac{\partial z}{\partial \theta}\frac{\partial x}{\partial \phi} - \frac{\partial x}{\partial \theta}\frac{\partial z}{\partial \phi}\right) + z\left(\frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \phi}\right)\right]\frac{d\theta \wedge d\phi}{R^3} \\ &= \sin\theta\, d\theta \wedge d\phi.\end{aligned}$$

This computation looks just like a change of variables although it is conceptually different.

(d) Using the pullback of the map

$$\begin{aligned}\varphi &: S^2 \to \mathbb{R}^3 \\ &: (\theta, \phi) \mapsto (x, y, z)\end{aligned}$$

computed in part (c), one finds

$$\int_{S^2(R)} \omega = \int_{\varphi^{-1}(S^2(R))} \varphi^*\omega = \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta d\phi = 4\pi.$$

Note that we cannot use Stokes' theorem here to write $\int_{S^2(R)} \omega = \int_{B^3(R)} d\omega = 0$ (we found $d\omega = 0$ in part (a)) since the 2-form ω is not smooth in the vicinity around the origin.

2 Sphere Area

Before determining the form for the surface "volume" of the n-sphere, we can note that the surface volume in the neighborhood of $\hat{\mathbf{e}}_k$ is given by

$$\omega(\hat{\mathbf{e}}_k) = (-1)^{k+1} x^k\, dx^1 \wedge \cdots \wedge dx^{k-1} \wedge dx^{k+1} \wedge \cdots \wedge dx^{n+1}.$$

This is just the volume form for the tangent space at $\hat{\mathbf{e}}_k$. The factor of $(-1)^k$ is found by requiring integration over a small box sitting at $\hat{\mathbf{e}}_k$ be positive. Now, the form for the surface "volume" of the n-sphere should reduce to this in the neighborhood of each $\hat{\mathbf{e}}_k$ and should be invariant under orthogonal transformations. One can easily see that the form that satisfies these conditions is just

$$\omega = \frac{1}{n!}\epsilon_{\alpha_1\alpha_2\ldots\alpha_{n+1}} x^{\alpha_1}\, dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_{n+1}}.$$

Taking the exterior derivative, one finds

$$\begin{aligned}d\omega &= d\left(\frac{1}{n!}\epsilon_{\alpha_1\alpha_2\ldots\alpha_{n+1}} x^{\alpha_1}\, dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_{n+1}}\right) \\ &= \frac{1}{n!}\epsilon_{\alpha_1\alpha_2\ldots\alpha_{n+1}}\, dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_{n+1}}\end{aligned}$$

$$= \underbrace{\frac{1}{n!}(n+1)!}_{=(n+1)}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{n+1}.$$

Using Stokes' Theorem, we find

$$\int_{S^n} \omega = \int_{B^{n+1}} d\omega = (n+1) \int_{B^{n+1}} d^{n+1}x,$$

which implies $\mathrm{Vol}(S^n)/\,\mathrm{Vol}(B^{n+1}) = n+1$, as desired.

3 Push and Pull

Let $\varphi : M \to N$ and $\omega \in \bigwedge^p T^*N$. We require φ to be invertible so that the pushforward is well-defined. As a convention, we use the local coordinates $\{x^i\}$ for M and $\{y^i\}$ for N. Recall the definitions for the pullback and pushforward (equations (12.32) and (12.29), respectively, from the textbook):

$$\varphi^*\omega = \frac{1}{p!}\omega_{\nu_1\ldots\nu_p}\frac{\partial y^{\nu_1}}{\partial x^{\mu_1}}\cdots\frac{\partial y^{\nu_p}}{\partial x^{\mu_p}}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} \qquad \text{(pullback)} \qquad (3)$$

$$\varphi_* X = X^\nu \frac{\partial y^\mu}{\partial x^\nu}\frac{\partial}{\partial y^\mu}. \qquad \text{(pushforward)} \qquad (4)$$

(a) Taking the exterior derivative of the pullback,

$$\begin{aligned}
d(\varphi^*\omega) &= \frac{1}{p!}d\left[\omega_{\nu_1\ldots\nu_p}\frac{dy^{\nu_1}}{dx^{\mu_1}}\cdots\frac{dy^{\nu_p}}{dx^{\mu_p}}\right]\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} \\
&= \frac{1}{p!}\left[\frac{\partial}{\partial x^{\mu_0}}\omega_{\nu_1\ldots\nu_p}\frac{dy^{\nu_1}}{dx^{\mu_1}}\cdots\frac{dy^{\nu_p}}{dx^{\mu_p}}\right]dx^{\mu_0}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} \\
&= \frac{1}{p!}\left[\frac{\partial}{\partial x^{\mu_0}}\omega_{\nu_1\ldots\nu_p}\right]\frac{dy^{\nu_1}}{dx^{\mu_1}}\cdots\frac{dy^{\nu_p}}{dx^{\mu_p}}dx^{\mu_0}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} \qquad \text{(product rule)} \\
&\quad + \frac{1}{p!}\omega_{\nu_1\ldots\nu_p}\underbrace{\left[\frac{\partial}{\partial x^{\mu_0}}\frac{dy^{\nu_1}}{dx^{\mu_1}}\cdots\frac{dy^{\nu_p}}{dx^{\mu_p}}\right]dx^{\mu_0}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}}_{=0},
\end{aligned}$$

where, in the last line, the second term vanishes since it is the contraction of a symmetric and antisymmetric tensor. What remains is the definition of $\varphi^*(d\omega)$.

(b) Recall that interior multiplication acts as a contraction with the coordinate representation of the form; i.e.,

$$i_X(\omega) = \left[\frac{1}{p!}\omega_{\mu_1\ldots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\right]\left(X^\sigma\frac{\partial}{\partial x^\sigma},\cdots\right) = \frac{1}{(p-1)!}\omega_{\sigma\mu_2\ldots\mu_p}X^\sigma dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_p}. \qquad (5)$$

In order to show $\mathcal{L}_X[\varphi^*\omega] = \varphi^*\mathcal{L}_{\varphi_* X}(\omega)$, we first work out how the pullback behaves with interior multiplication. Using equations (3) and (5) and local coordinates, one finds

$$\begin{aligned}
i_X(\varphi^*\omega) &= \frac{1}{(p-1)!}\omega_{\nu_1\dots\nu_p}\frac{\partial y^{\nu_1}}{\partial x^\sigma}\frac{\partial y^{\nu_2}}{\partial x^{\mu_2}}\cdots\frac{\partial y^{\nu_p}}{\partial x^{\mu_p}}X^\sigma dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_p} \\
&= \frac{1}{(p-1)!}\omega_{\nu_1\dots\nu_p}\frac{\partial y^{\nu_2}}{\partial x^{\mu_2}}\cdots\frac{\partial y^{\nu_p}}{\partial x^{\mu_p}}\underbrace{\left(\frac{\partial y^{\nu_1}}{\partial x^\sigma}X^\sigma\right)}_{=(\varphi_*X)^{\nu_1}} dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_p} = \varphi^*(i_{\varphi_*X}\omega).
\end{aligned}$$

The result follows by using the infinitesimal homotopy relation,

$$\mathcal{L}_X(\omega) = i_X d\omega + d\left(i_X\omega\right), \tag{6}$$

and result of the previous part, $\varphi^*(d\omega) = d(\varphi^*\omega)$. Plugging the pullback of ω into (6),

$$\mathcal{L}_X(\varphi^*\omega) = i_X d(\varphi^*\omega) + d\left(i_X(\varphi^*\omega)\right) = \varphi^*\left[i_{\varphi_*X}d\omega + d(i_{\varphi_*X}\omega)\right] = \varphi^*\mathcal{L}_{\varphi_*X}(\omega),$$

as desired.

(c) Starting with the definition of the Lie bracket,

$$[X,Y] = \left[X^\mu\left(\frac{\partial}{\partial x^\mu}Y^\nu\right) - Y^\mu\left(\frac{\partial}{\partial x^\mu}X^\nu\right)\right]\frac{\partial}{\partial x^\nu}, \tag{7}$$

one can plug in the pushforward of the vector fields and calculate the result directly.

$$\begin{aligned}
[\varphi_*X,\varphi_*Y]^\nu &= (\varphi_*X)^\mu\left(\frac{\partial}{\partial y^\mu}(\varphi_*Y)^\nu\right) - (\varphi_*Y)^\mu\left(\frac{\partial}{\partial y^\mu}(\varphi_*X)^\nu\right) \\
&= \left(\frac{\partial y^\mu}{\partial x^\lambda}X^\lambda\right)\left[\frac{\partial}{\partial y^\mu}\left(\frac{\partial y^\nu}{\partial x^\sigma}X^\sigma\right)\right] - \left(\frac{\partial y^\mu}{\partial x^\lambda}Y^\lambda\right)\left[\frac{\partial}{\partial y^\mu}\left(\frac{\partial y^\nu}{\partial x^\sigma}X^\sigma\right)\right] \\
&= X^\lambda\underbrace{\left(\frac{\partial y^\mu}{\partial x^\lambda}\frac{\partial}{\partial y^\mu}\right)}_{=\frac{\partial}{\partial x^\lambda}}\left(\frac{\partial y^\nu}{\partial x^\sigma}Y^\sigma\right) - Y^\lambda\underbrace{\left(\frac{\partial y^\mu}{\partial x^\lambda}\frac{\partial}{\partial y^\mu}\right)}_{=\frac{\partial}{\partial x^\lambda}}\left(\frac{\partial y^\nu}{\partial x^\sigma}X^\sigma\right) && \text{(chain rule)} \\
&= \cancel{X^\lambda\left[\frac{\partial}{\partial x^\sigma}\frac{\partial y^\nu}{\partial x^\sigma}\right]}Y^\sigma + \left(\frac{\partial y^\nu}{\partial x^\sigma}\right)X^\lambda\left(\frac{\partial}{\partial x^\lambda}Y^\sigma\right) && (\partial_\mu\partial_\nu = \partial_\nu\partial_\mu) \\
&\quad \cancel{-Y^\lambda\left[\frac{\partial}{\partial x^\sigma}\frac{\partial y^\nu}{\partial x^\sigma}\right]}X^\sigma + \left(\frac{\partial y^\nu}{\partial x^\sigma}\right)Y^\lambda\left(\frac{\partial}{\partial x^\lambda}X^\sigma\right) \\
&= \left(\frac{\partial y^\nu}{\partial x^\sigma}\right)[X,Y]^\sigma \\
&= (\varphi_*[X,Y])^\nu.
\end{aligned}$$

4 Stereographic Coordinates

For this problem, it is convenient to organize our calculation using matrix notation. We let $g^{(S^2)}$ denote the metric on S^2, which is given by

$$g^{(S^2)}_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix}_{\mu\nu}.$$

we are given the map

$$\begin{aligned} \zeta &: S^2 \to \mathbb{C} \\ &: (\theta, \phi) \mapsto \zeta = e^{i\phi}\tan(\theta/2), \end{aligned}$$

and want to compute the new metric on $\mathbb{C}$, $g^{(\mathbb{C})}$. Using the transformation properties for doubly covariant tensors (i.e., $(0,2)$-tensors), we can write

$$g^{(\mathbb{C})}_{\alpha\beta} = \frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta}, \qquad FIXME$$

where $y^1 = \theta$ and $y^2 = \phi$ denote the coordinates on S^2 and $x^1 = \zeta$ and $x^2 = \bar{\zeta}$ denote the coordinates on $\mathbb{C}$. Since we are given ζ as a function of θ and ϕ in the problem, it is easier to transform in the reverse direction and then invert the resulting relationship; i.e.,

$$g^{(\mathbb{C})}_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu} = g^{(S^2)}_{\mu\nu}.$$

In matrix form, this corresponds to[4]

$$J^t g^{(\mathbb{C})} J = g^{(S^2)},$$

where the Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial\zeta}{\partial\theta} & \frac{\partial\zeta}{\partial\phi} \\ \frac{\partial\bar{\zeta}}{\partial\theta} & \frac{\partial\bar{\zeta}}{\partial\phi} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{i\phi}\sec^2\left(\frac{\theta}{2}\right) & ie^{i\phi}\tan\left(\frac{\theta}{2}\right) \\ \frac{1}{2}e^{-i\phi}\sec^2\left(\frac{\theta}{2}\right) & -ie^{-i\phi}\tan\left(\frac{\theta}{2}\right) \end{pmatrix}.$$

Inverting matrices and going through a bit of arithmetic, one finds

$$g^{(\mathbb{C})}_{\alpha\beta} = \frac{2}{\left(1+|\zeta|^2\right)^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\alpha\beta},$$

[4] Be careful here with the indices. The transpose here is necessary so that the metric tensor transforms like a $(0,2)$-tensor (like a quadratic form) rather than a $(1,1)$-tensor. See section 10.2.2 in the textbook.

or equivalently (in the original notation used in the problem),[5]

$$g(\,\cdot\,,\,\cdot\,) = \frac{2}{\left(1+|\zeta|^2\right)^2}\left(d\zeta\otimes d\bar{\zeta} + d\bar{\zeta}\otimes d\zeta\right).$$

The final equality stated in the problem follows immediately since

$$d\zeta\otimes d\bar{\zeta} = (d\xi + id\eta)\otimes(d\xi + id\eta) = d\xi\otimes d\xi + d\eta\otimes d\eta - id\xi\otimes d\eta + id\eta\otimes d\xi.$$

Hence $d\zeta\otimes d\bar{\zeta} + d\bar{\zeta}\otimes d\zeta = 2\,(d\xi\otimes d\xi + d\eta\otimes d\eta)$, which shows that

$$g(\,\cdot\,,\,\cdot\,) = \frac{4}{\left(1+|\xi|^2+|\eta|^2\right)^2}\left(d\xi\otimes d\xi + d\eta\otimes d\eta\right),$$

as desired.

Finally, note that the corresponding volume forms are just given by[6]

$$\Omega = \sqrt{g}\,d\bar{\zeta}\wedge d\zeta = \frac{2i}{\left(1+|\zeta|^2\right)^2}\,d\bar{\zeta}\wedge d\zeta.$$

Or, in terms of the coordinates ξ and η,

$$\Omega = \sqrt{g}\,d\xi\wedge d\eta = \frac{4}{\left(1+|\xi|^2+|\eta|^2\right)^2}\,d\xi\wedge d\eta.$$

5 Bogomolnyi Equations

(a) Letting $y^1 = \xi$ and $y^2 = \eta$, we can write the winding number N in terms of $\{y^\mu\}$ simply by pulling back the volume form (found in the previous question) by the spin field map, $n : x \mapsto \hat{\mathbf{n}}(x)$.[7]

$$n^*\Omega = \frac{1}{2!}\Omega_{\mu\nu}\frac{\partial y^\mu}{\partial x^\gamma}\frac{\partial y^\nu}{\partial x^\delta}\,dx^\gamma\wedge dx^\delta = \Omega_{\xi\eta}\left(\frac{\partial\xi}{\partial x^1}\frac{\partial\eta}{\partial x^2} - \frac{\partial\eta}{\partial x^1}\frac{\partial\xi}{\partial x^2}\right)dx^1\wedge dx^2.$$

Hence, the winding number is

$$4\pi N = \int n^*\Omega = \int\frac{4}{\left(1+|\xi|^2+|\eta|^2\right)^2}\left[\left(\frac{\partial\xi}{\partial x^1}\right)\left(\frac{\partial\eta}{\partial x^2}\right) - \left(\frac{\partial\eta}{\partial x^1}\right)\left(\frac{\partial\xi}{\partial x^2}\right)\right]dx^1\wedge dx^2.$$

[5] I'll drop the superscript labeling the metric tensors.
[6] See "Volume Form" subsection in 12.2 of the textbook.
[7] By a slight abuse of notation, I write $\Omega_{12} = \Omega_{y^1y^2} = \Omega_{\xi\eta}$.

To write the energy functional in the desired form, first note that

$$E[\hat{\mathbf{n}}] = \frac{1}{2}\int \left(\left|\nabla n^1\right|^2 + \left|\nabla n^2\right|^2 + \left|\nabla n^3\right|^2\right)\, dx^1 \wedge dx^2 = \frac{1}{2}\int g^{(\mathbb{R}^2)}_{\alpha\beta}\delta^{\gamma\delta}\frac{\partial n^\alpha}{\partial x^\gamma}\frac{\partial n^\beta}{\partial x^\delta}\, dx^1 \wedge dx^2,$$

where $g^{(\mathbb{R}^2)}_{\alpha\beta}$ denotes the metric on $\mathbb{R}^2 \simeq \mathbb{C}$. We then pull back the spin field and write the remaining bits as the metric on S^2 (from the previous problem),

$$g^{(\mathbb{R}^2)}_{\alpha\beta}\delta^{\gamma\delta}\frac{\partial n^\alpha}{\partial x^\gamma}\frac{\partial n^\beta}{\partial x^\delta} = g^{(\mathbb{R}^2)}_{\alpha\beta}\delta^{\gamma\delta}\left(\frac{\partial n^\alpha}{\partial y^\sigma}\frac{\partial y^\sigma}{\partial x^\gamma}\right)\left(\frac{\partial n^\beta}{\partial y^\rho}\frac{\partial y^\rho}{\partial x^\delta}\right) = \underbrace{\left(g^{(\mathbb{R}^2)}_{\alpha\beta}\frac{\partial n^\alpha}{\partial y^\sigma}\frac{\partial n^\beta}{\partial y^\rho}\right)}_{=g^{(S^2)}_{\sigma\rho}}\frac{\partial y^\sigma}{\partial x^\gamma}\frac{\partial y^\rho}{\partial x^\delta}.$$

Plugging this into the energy functional yields the desired result,

$$E[\hat{\mathbf{n}}] = \frac{1}{2}\int \frac{4}{\left(1+|\xi|^2+|\eta|^2\right)^2}\left[\left(\frac{\partial\xi}{\partial x^1}\right)^2 + \left(\frac{\partial\xi}{\partial x^2}\right)^2 + \left(\frac{\partial\eta}{\partial x^1}\right)^2 + \left(\frac{\partial\eta}{\partial x^2}\right)^2\right] dx^1 \wedge dx^2.$$

(b) Using $\partial_i \equiv \frac{\partial}{\partial x^i}$, note that

$$|(\partial_1 + i\partial_2)(\xi + i\eta)|^2 = (\partial_1\xi)^2 + (\partial_2\xi)^2 + (\partial_1\eta)^2 + (\partial_2\eta)^2 - (\partial_1\xi)(\partial_2\eta) + (\partial_1\eta)(\partial_2\xi).$$

Hence,

$$E - 4\pi N = \frac{1}{2}\int \frac{4}{\left(1+|\xi|^2+|\eta|^2\right)^2}\,|(\partial_1 + i\partial_2)(\xi + i\eta)|^2\; dx^1 \wedge dx^2 \geq 0$$

since the integrand is non-negative.

(c) If $N > 0$, then minimizing the energy immediately shows that $|(\partial_1 + i\partial_2)(\xi + i\eta)|^2 = 0 \implies (\partial_1 + i\partial_2)(\xi + i\eta) = 0$. This $\xi + i\eta$ is meromorphic in the entire complex plane and hence a rational function.

(d) Since one is not expected to be familiar with complex analysis at this stage in the course, we simply check that the proposed solution,

$$\xi + i\eta = \lambda\frac{(z-a_1)\cdots(z-a_N)}{(z-b_1)\cdots(z-b_N)},$$

satisfies the condition. But this is easy since $(\partial_1 + i\partial_2) = \partial_{\bar{z}}$, and $\xi + i\eta$ has been written independently of $\bar{z}$. Hence $\partial_{\bar{z}}(\xi + i\eta) = 0$, as desired.

(e) Now, for $N < 0$, the quantity $E - 4\pi N$ is not much use to us; instead, consider the quantity $E + 4\pi N$. Following analogous steps as those in parts (b) through (d), one finds

$$E + 4\pi N = \frac{1}{2}\int \frac{4}{\left(1+|\xi|^2+|\eta|^2\right)^2}\,|(\partial_1 - i\partial_2)(\xi + i\eta)|^2\; dx^1 \wedge dx^2 \geq 0,$$

so that the minimum energy corresponds to $E = 4\pi \, |N|$. Additionally, the function $\xi + i\eta$ is now anti-meromorphic in the entire complex plane; i.e.,

$$(\partial_1 - i\partial_2)(\xi + i\eta) = \partial_z(\xi + i\eta) = 0,$$

and so we can write

$$\xi + i\eta = \lambda \frac{(\bar{z} - a_1) \cdots (\bar{z} - a_N)}{(\bar{z} - b_1) \cdots (\bar{z} - b_N)},$$

as desired.