Physics 509 Homework 2 | Professor M. Stone
Spring 2022 | University of Illinois

1 Lie Bracket Geometry

Recall the definition of the flow associated with a tangent-vector field (equation (11.24) in the textbook): it is the map that takes a point x_0 and maps it to $x(t)$ by solving the family of equations

$$\frac{dx^\mu}{dt} = X^\mu(x^1, x^2, \dots), \tag{1}$$

with initial condition $x^\mu(0) = x_0^\mu$. The resulting (differential) equations are found easily enough.

$$X = y\partial_x \implies \begin{cases} \dot{x} = y \\ \dot{y} = 0 \end{cases} \implies \begin{cases} x(t) = y_0 t + x_0 \\ y(t) = y_0 \end{cases},$$

and likewise for Y,

$$Y = \partial_y \implies \begin{cases} \dot{x} = 0 \\ \dot{y} = 1 \end{cases} \implies \begin{cases} x(t) = x_0 \\ y(t) = t + y_0 \end{cases}.$$

Hence the flows associated with X and Y are

$$\Phi^X(t) = (y_0 t + x_0, y_0) \qquad \text{and} \qquad \Phi^Y(t) = (x_0, t + y_0).$$

The commutator is easily calculated:

$$\begin{aligned} [X, Y] &= XY - YX \\ &= y\partial_x\partial_y - \partial_y(y\partial_x) \\ &= \cancel{y\partial_x\partial_y} - \cancel{y\partial_x\partial_y} - \partial_x \\ &= -\partial_x. \end{aligned}$$

The geometric interpretation of the Lie bracket is discussed in section 11.2 of the textbook (see figure 11.3). Figure 1 shows this geometric interpretation for the case of the vector fields X and Y.

2 Frobenius' Theorem

Remember that a set of vector fields, $\{X_i\}$, are said to be in involution with each other if the Lie bracket is closed; i.e.,

$$[X_i, X_j] = {c_{ij}}^k X_k, \tag{2}$$

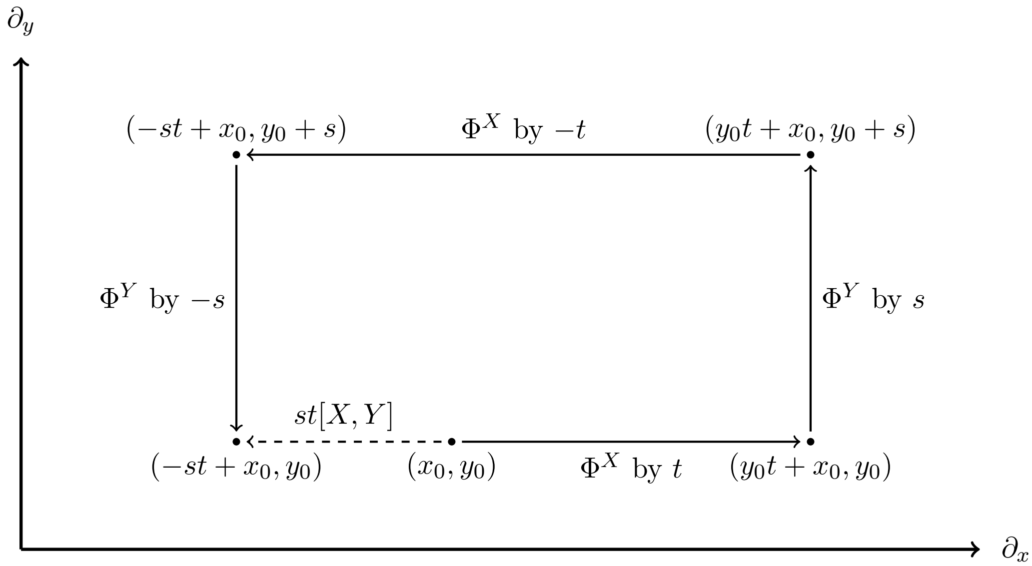


Figure 1: The trajectory of an initial point (x_0, y_0) as it first flows along X by t, then along Y by s, and - starting the trip back - along X by $-t$, and finally along Y by $-s$. In agreement with the geometric interpretation discussed in the textbook, the difference between the initial and final points (dashed line) is just $st[X, Y] = -st\partial_x$.

for some set of functions ${c_{ij}}^k$. By direct calculation (or in analogy with angular momentum),

$$\begin{aligned}
[L_y, L_z] &= [z\partial_x - x\partial_z, x\partial_y - y\partial_x] \\
&= [z\partial_x, x\partial_y] - \cancel{[z\partial_x, y\partial_x]} - \cancel{[x\partial_z, x\partial_y]} + [x\partial_z, y\partial_x] && \text{(linearity of } [\cdot,\cdot]) \\
&= z\partial_y - y\partial_z && (= -L_x) \\
&= \frac{z}{x}\underbrace{(x\partial_y - y\partial_x)}_{=L_z} + \frac{y}{x}\underbrace{(z\partial_x - x\partial_z)}_{=L_y}.
\end{aligned}$$

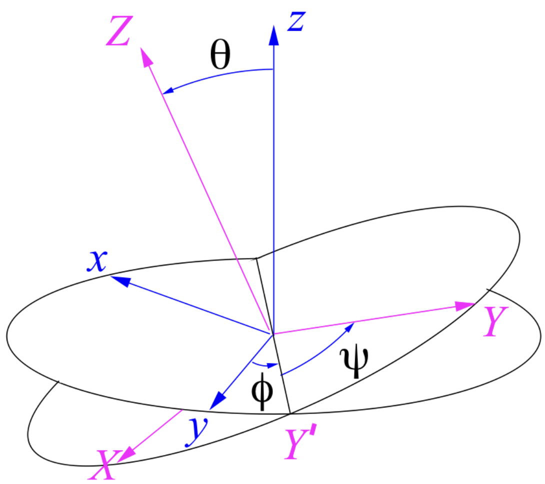
This shows $c^z_{yz} = z/x$ and $c^y_{yz} = y/x$, which satisfies definition (2), and hence L_y and L_z are in involution. If

$$L_z f = (x\partial_y - y\partial_x)f = 0 \qquad \text{and} \qquad -L_y f = (x\partial_z - z\partial_x)f = 0,$$

then $L^2 f \equiv (L_x^2 + L_y^2 + L_z^2)f = 0$ as well, since L_y and L_z are in involution. In other words, f is an eigenfunction of the "total angular momentum" operator and therefore must be spherically symmetric.

3 Rolling Ball

Expressions for $(\omega_x, \omega_y, \omega_z)$ in terms of the Euler angles can be read off directly from the diagram (given in the problem).



$$\omega_x = \underbrace{-\dot\psi\sin\theta\cos\phi}_{\substack{\text{project } \dot\psi \text{ to}\\ yz\text{-plane}}} + \underbrace{\dot\theta\sin\phi}_{\substack{\text{project } \dot\theta \text{ to}\\ yz\text{-plane}}}, \tag{3a}$$

$$\omega_y = \underbrace{\dot\psi\sin\theta\sin\phi}_{\substack{\text{project } \dot\psi \text{ to}\\ xz\text{-plane}}} + \underbrace{\dot\theta\cos\phi}_{\substack{\text{project } \dot\theta \text{ to}\\ xz\text{-plane}}}, \tag{3b}$$

$$\omega_z = \underbrace{\dot\psi\cos\theta}_{\substack{\text{project } \dot\psi \text{ to}\\ xy\text{-plane}}} + \underbrace{\dot\phi}_{\substack{\text{already in}\\ xy\text{-plane}}} \tag{3c}$$

For example, to find ω_z we project all the angular velocities (i.e., $\dot\theta$, $\dot\psi$, $\dot\phi$) to the plane perpendicular to the z-axis. This procedure yields equations (3a) through (3c).

(a) The rolling conditions for a ball on a table mentioned in class are

$$\begin{aligned}\dot x &= \dot\psi\sin\theta\sin\phi + \dot\theta\cos\phi,\\ \dot y &= -\dot\psi\sin\theta\cos\phi + \dot\theta\sin\phi,\\ 0 &= \dot\psi\cos\theta + \dot\phi.\end{aligned}$$

Comparison with equations (3a), (3b), and (3c) shows that these are identically the no-slip rolling conditions:

$$\begin{aligned}\dot x &= \omega_y,\\ \dot y &= -\omega_x,\\ 0 &= \omega_z.\end{aligned}$$

(b) This problem can, essentially, be thought of as the reverse of problem one where we were given a vector field and asked to find its associated flow. Here, we are gifted a system of differential equations that specify a flow and asked to solve for the associated vector field. In the case of $\mathbf{roll_x}$, it is a flow of unit velocity along the x-direction specified by $\dot x = 1$, $\dot y = 0$, and the three no-slip conditions, which give a system of five equations in five unknowns (the dotted variables). The case of $\mathbf{roll_y}$ is analogous.

To find the vector field $\mathbf{roll_x}$ associated to the flow of unit speed along the x direction, we set

$\dot{x} = 1$, $\dot{y} = 0$, and then solve the resulting system of equations,

$$1 = \dot{\psi} \sin\theta \sin\phi + \dot{\theta} \cos\phi \tag{4a}$$
$$0 = -\dot{\psi} \sin\theta \cos\phi + \dot{\theta} \sin\phi \tag{4b}$$
$$0 = \dot{\psi} \cos\theta + \dot{\phi}, \tag{4c}$$

in terms of the local coordinates (here, only ϕ, θ, and ψ). A little bit of arithmetic yields

$$\text{(4b)} \tan\phi + \text{(4a)} \implies \dot{\theta} = \cos\phi \tag{5a}$$
$$\text{(4b)}(-\cot\phi) + \text{(4a)} \implies \dot{\psi} = \sin\phi \csc\theta. \tag{5b}$$

Lastly, substituting equation (5c) into equation (4c) gives

$$\dot{\phi} = -\dot{\phi} \cos\theta = -\left(\frac{\dot{x} \sin\phi}{\sin\theta}\right) \cos\theta \implies \dot{\phi} = -\sin\phi \cot\theta. \tag{5c}$$

Using (5a) through (5c), one can write the vector field in terms of the local coordinates,

$$\mathbf{roll_x} = \left(\dot{x}(t), \dot{y}(t), \dot{\phi}(t), \dot{\theta}(t), \dot{\psi}(t)\right) = \partial_x - (\sin\phi \cot\theta)\partial_\phi + (\cos\phi)\partial_\theta + (\sin\phi \csc\theta)\partial_\psi. \tag{6}$$

In the second equality, I've have written the vector field in terms of the (tangent space) basis elements $\{\partial_x, \partial_y, \partial_\phi, \partial_\theta, \partial_\psi\}$.

We use the analogous procedure to find Y, which instead satisfies the conditions $\dot{x} = 0$ and $\dot{y} = 1$. This gives the following system of equations:

$$0 = \dot{\psi} \sin\theta \sin\phi + \dot{\theta} \cos\phi \tag{7a}$$
$$1 = -\dot{\psi} \sin\theta \cos\phi + \dot{\theta} \sin\phi \tag{7b}$$
$$0 = \dot{\psi} \cos\theta + \dot{\phi}. \tag{7c}$$

These again are solved easily.

$$\text{(7a)} \cot\phi + \text{(7b)} \implies \dot{\theta} = \sin\phi \tag{8a}$$
$$\text{(7a)}(-\tan\phi) + \text{(7b)} \implies \dot{\psi} = -\csc\theta \cos\phi. \tag{8b}$$

Finally, substituting (8b) into (7c), we find

$$\dot{\phi} = -\dot{\psi} \cos\theta = \cos\phi \cot\theta \implies \dot{\phi} = \cos\phi \cot\theta. \tag{8c}$$

Using equations (8a) through (8c), one finds

$$\mathbf{roll_y} = \partial_y + (\cos\phi \cot\theta)\partial_\phi + (\sin\phi)\partial_\theta - (\csc\theta \cos\phi)\partial_\psi. \tag{9}$$

(c) Having found both vector fields $\mathbf{roll_x}$ and $\mathbf{roll_y}$, we can compute the commutator by direct calculation.

$$
\begin{aligned}
[\mathbf{roll_x}, \mathbf{roll_y}] &= (\mathbf{roll_x})(\mathbf{roll_y}) - (\mathbf{roll_y})(\mathbf{roll_x}) \\
&= \ (\partial_x - (\sin\phi\cot\theta)\partial_\phi + (\cos\phi)\partial_\theta + (\sin\phi\csc\theta)\partial_\psi) \\
&\quad\ (\partial_y + (\cos\phi\cot\theta)\partial_\phi + (\sin\phi)\partial_\theta - (\csc\theta\cos\phi)\partial_\psi) \\
&\quad - (\partial_y + (\cos\phi\cot\theta)\partial_\phi + (\sin\phi)\partial_\theta - (\csc\theta\cos\phi)\partial_\psi) \\
&\quad\quad (\partial_x - (\sin\phi\cot\theta)\partial_\phi + (\cos\phi)\partial_\theta + (\sin\phi\csc\theta)\partial_\psi) \\
&\hspace{18em} \text{(plug in (6) and (9))} \\
&= \ \big((\sin^2\phi\cot^2\theta)\partial_\phi - (\sin\phi\cos\phi\cot\theta)\partial_\theta - (\csc\theta\cot\theta\sin^2\phi)\partial_\psi \\
&\qquad\qquad - (\cos^2\phi\csc^2\theta)\partial_\phi + (\cot\phi\csc\phi\cos^2\phi)\partial_\psi\big) \\
&\quad - \big(-(\cos^2\phi\cot^2\theta)\partial_\phi - (\sin\phi\cos\phi\cot\theta)\partial_\theta + (\cos^2\phi\csc\theta\cot\theta)\partial_\psi \\
&\qquad\qquad + (\sin^2\phi\csc^2\theta)\partial_\phi - (\sin^2\phi\cot\theta\csc\theta)\partial_\psi\big) \\
&= \underbrace{(\cot^2\theta - \csc^2\theta)}_{=-1}\partial_\phi.
\end{aligned}
$$

Since $\mathbf{spin_z} = \partial_\phi$, we arrive at the desired result,

$$
[\mathbf{roll_x}, \mathbf{roll_y}] = -\,\mathbf{spin_z}\,. \tag{10}
$$

(d) The commutators can be computed directly:

$$
\begin{aligned}
[\mathbf{spin_z}, \mathbf{roll_x}] &= (\mathbf{spin_z})(\mathbf{roll_x}) - (\mathbf{roll_x})(\mathbf{spin_z}) \\
&= -(\cos\phi\cot\theta)\partial_\phi - (\sin\phi)\partial_\theta + (\csc\theta\cos\phi)\partial_\psi = -(\mathbf{roll_y} - \partial_y) \equiv \mathbf{spin_x}
\end{aligned}
$$

and

$$
\begin{aligned}
[\mathbf{spin_z}, \mathbf{roll_y}] &= (\mathbf{spin_z})(\mathbf{roll_y}) - (\mathbf{roll_y})(\mathbf{spin_z}) \\
&= -(\sin\phi\cot\theta)\partial_\phi + (\cos\theta)\partial_\theta + (\csc\theta\sin\phi)\partial_\psi = (\mathbf{roll_x} - \partial_x) \equiv \mathbf{spin_y}\,.
\end{aligned}
$$

Note that we have generated five linearly independent vector fields by taking commutators of $\mathbf{roll_x}$ and $\mathbf{roll_y}$. This shows in fact that any point on the manifold can be reached only by rolling in the x or y direction.

4 Killing Vector

Using equation (11.38) from the textbook for the Lie derivative of a type $(0,2)$ tensor,

$$(\mathcal{L}_X\, g)_{\mu\nu} = X^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha}\partial_\nu X^\alpha + g_{\alpha\nu}\partial_\mu X^\alpha, \tag{11}$$

we can check by direct computation that $\mathcal{L}_{V_x}\, g = 0$, where

$$g(\ ,\ ) = d\theta \otimes d\theta + \sin^2(\theta) d\phi \otimes d\phi$$

and

$$V_x = -\sin(\phi)\partial_\theta - \cot(\theta)\cos(\phi)\partial_\phi.$$

We first calculate all the necessary derivatives.

$$\partial_\theta g_{\phi\phi} = 2\sin(\theta)\cos(\theta) \qquad \partial_\theta V_x^\phi = \csc^2(\theta)\cos(\phi)$$
$$\partial_\phi V_x^\theta = -\cos(\phi) \qquad \partial_\phi V_x^\phi = \cot(\theta)\sin(\phi)$$

All other derivatives vanish. Next, just show that all the components vanish identically:

$$\begin{aligned}
(\mathcal{L}_{V_x}\, g)_{\phi\phi} = &\ V_x^\phi \partial_\phi g_{\phi\phi} + V_x^\theta \partial_\theta g_{\phi\phi} + g_{\phi\phi}\partial_\phi V_x^\phi \\
&+ g_{\phi\theta}\partial_\phi V_x^\theta + g_{\phi\phi}\partial_\phi V_x^\phi + g_{\theta\phi}\partial_\phi V_x^\theta \\
= &\ (-\sin(\phi))(2\sin(\theta)\cos(\theta)) + (\sin^2(\theta))(\cot(\theta)\sin(\phi)) \\
&\qquad + (\sin^2(\theta))(\cot(\theta)\sin(\phi)) \\
= &\ 0,
\end{aligned}$$

$$\begin{aligned}
(\mathcal{L}_{V_x}\, g)_{\theta\theta} = &\ V_x^\phi \partial_\phi g_{\theta\theta} + V_x^\theta \partial_\theta g_{\theta\theta} + g_{\theta\phi}\partial_\theta V_x^\phi \\
&+ g_{\theta\theta}\partial_\theta V_x^\theta + g_{\phi\theta}\partial_\theta V_x^\phi + g_{\theta\theta}\partial_\theta V_x^\theta \\
= &\ 0, \qquad \text{(all terms multiplied by zero)}
\end{aligned}$$

$$\begin{aligned}
(\mathcal{L}_{V_x}\, g)_{\phi\theta} = &\ V_x^\phi \partial_\phi g_{\phi\theta} + V_x^\theta \partial_\theta g_{\phi\theta} + g_{\phi\phi}\partial_\theta V_x^\phi \\
&+ g_{\phi\theta}\partial_\theta V_x^\theta + g_{\phi\theta}\partial_\phi V_x^\phi + g_{\theta\theta}\partial_\phi V_x^\theta \\
= &\ (\sin^2(\theta))(\csc^2(\theta)\cos(\phi)) - \cos(\phi) \\
= &\ 0,
\end{aligned}$$

$$
\begin{aligned}
(\mathcal{L}_{V_x}\, g)_{\theta\phi} = {} & V_x^{\phi}\partial_\phi g_{\phi\phi} + V_x^{\theta}\partial_\theta g_{\phi\phi} + g_{\phi\phi}\partial_\phi V_x^{\phi} \\
& + g_{\phi\theta}\partial_\phi V_x^{\theta} + g_{\phi\phi}\partial_\phi V_x^{\phi} + g_{\theta\phi}\partial_\phi V_x^{\theta} \\
= {} & -\cos(\phi) + (\sin^2(\theta))(\csc^2(\theta)\cos(\phi)) \\
= {} & 0.
\end{aligned}
$$

Hence, $\mathcal{L}_{V_x}\, g = 0$ as expected.