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# Addition of Angular Momenta

Consider first a simple problem: two distinguishable particles each of spin 1/2. We want to know ~~how~~ to classify their states. Clearly we have $\vec{S}_1$ and $\vec{S}_2$, with each ~~$\vec{S}_1^2 / \frac{1}{2}\hbar$~~, just two states $|\uparrow\rangle_1$, $|\downarrow\rangle_1$, $|\uparrow\rangle_2$, $|\downarrow\rangle_2$

$\Rightarrow$ we have a total of 4 states

$$|\uparrow\uparrow\rangle \;,\; |\uparrow\downarrow\rangle \;,\; |\downarrow\uparrow\rangle \;,\; |\downarrow\downarrow\rangle$$

$$S_1^z|\uparrow\uparrow\rangle = \frac{\hbar}{2}|\uparrow\uparrow\rangle \qquad S_1^z|\uparrow\downarrow\rangle = +\frac{\hbar}{2}|\uparrow\downarrow\rangle$$

$$S_1^z|\downarrow\uparrow\rangle = -\frac{\hbar}{2}|\downarrow\uparrow\rangle \qquad S_1^z|\downarrow\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\downarrow\rangle$$

and so on.

Let $\vec{S} = \vec{S}_1 + \vec{S}_2$

$$[S_1^i, S_2^j] = 0$$

$$\Rightarrow [S_i, S_j] = i\hbar\, \epsilon_{ijk} S_k$$

Q: What are the eigenstates ~~with~~ of $\vec{S}^2$ ? and of $S_z$?

$$S_z|\uparrow\uparrow\rangle = (S_{1z} + S_{2z})|\uparrow\uparrow\rangle = \left(\frac{\hbar}{2} + \frac{\hbar}{2}\right)|\uparrow\uparrow\rangle = \hbar|\uparrow\uparrow\rangle$$

$$S_z|\downarrow\downarrow\rangle = \left(-\frac{\hbar}{2} - \frac{\hbar}{2}\right)|\downarrow\downarrow\rangle = -\hbar|\downarrow\downarrow\rangle$$

$$S_z|\uparrow\downarrow\rangle = \left(\frac{\hbar}{2} - \frac{\hbar}{2}\right)|\uparrow\downarrow\rangle = 0$$

$$S_z|\downarrow\uparrow\rangle = \left(-\frac{\hbar}{2} + \frac{\hbar}{2}\right)|\downarrow\uparrow\rangle = 0$$

$$\vec{S}^2 = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + 2\,\vec{S}_1\cdot\vec{S}_2$$

$$\vec{S}_1\cdot\vec{S}_2 = S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}$$

$$= \frac{1}{2}(S_{1x} + iS_{1y})(S_{2x} - iS_{2y}) + \frac{1}{2}(S_{1x} - iS_{1y})(S_{2x} + iS_{2y}) + S_{1z}S_{1z}$$

$$= \frac{1}{2}(S_{1}^{+}S_{2}^{-} + S_1^{-}S_2^{+}) + S_{1z}S_{2z}$$

$$\Rightarrow \vec{S}^2 = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + 2\cdot\frac{1}{2}(S_1^+S_2^- + S_1^-S_2^+) + 2S_{1z}S_{2z}$$

$$\Rightarrow \vec{S}^2|\uparrow\uparrow\rangle = \hbar^2\frac{3}{4}|\uparrow\uparrow\rangle + \hbar^2\frac{3}{4}|\uparrow\uparrow\rangle + 0 + 0 + 2\left(\frac{\hbar}{2}\right)^2|\uparrow\uparrow\rangle$$

$$\vec{S}^2|s,m\rangle = \hbar^2 s(s+1)\,|s,m\rangle$$

$$S^+|s,s\rangle = 0 \qquad\qquad S^{\pm}|s,m\rangle = \hbar\sqrt{s(s+1) - m(m\pm1)}\;|s,m\pm1\rangle$$

$$S^-|s,-s\rangle = 0 \qquad\qquad = \hbar\sqrt{(s\mp m)(s\pm m+1)}\;|s,m\pm1\rangle$$

$$\Rightarrow \vec{S}^2|\uparrow\uparrow\rangle = \hbar^2\left(\frac{3}{4} + \frac{3}{4} + \frac{2}{4}\right)|\uparrow\uparrow\rangle = 2\hbar^2|\uparrow\uparrow\rangle$$

$\Rightarrow s = 1$ since $s(s+1) = 2$

$$S_z|\uparrow\uparrow\rangle = \hbar\,|\uparrow\uparrow\rangle$$

$$\Rightarrow |\uparrow\uparrow\rangle = |1\,1\rangle \qquad s=1,\ M=1$$

What is the state $|1\,0\rangle$ ?

$$S^-|s\,m\rangle = \hbar\sqrt{s(s+1) - m(m-1)}\;|s,m-1\rangle$$

$$\Rightarrow S^-|1\,1\rangle = \hbar\sqrt{2-0}\;|1,0\rangle$$

$$\Rightarrow |1,0\rangle = \frac{1}{\sqrt{2}\,\hbar}\,S^-|1\,1\rangle$$

$$S^- = S_1^- + S_2^-$$

$$S^- |1\,1\rangle = S_1^- |\uparrow\uparrow\rangle + S_2^- |\uparrow\uparrow\rangle = \hbar \left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right)$$

$$S_1^- |\uparrow\rangle = \hbar\ |\downarrow\rangle$$

$$S_1^+ |\downarrow\rangle = \hbar\ |\uparrow\rangle$$

But $\quad S^- |1\,1\rangle = \hbar\sqrt{1(1+1) - 1(1-1)}\ |1\,0\rangle$

$$\Rightarrow \quad |1\,0\rangle = \frac{1}{\sqrt{2}\hbar}\ S^- |1\,1\rangle$$

$$\Rightarrow \quad |1\,0\rangle = \frac{1}{\sqrt{2}\cancel{\hbar}}\ \cancel{\hbar} \left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right)$$

$$\Rightarrow \quad |1\,0\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

Similarly

$$|1,-1\rangle = \quad S_- |1\,0\rangle \quad \frac{1}{\hbar\sqrt{1.2 - 0.1}}$$

$$|1\,{-1}\rangle = \frac{1}{\hbar\sqrt{2}}\, S_- |1\,0\rangle$$

$$|1\,{-1}\rangle = \frac{1}{\hbar\sqrt{2}}\ \frac{1}{\sqrt{2}} \left[ S_- |\uparrow\downarrow\rangle + S_- |\downarrow\uparrow\rangle \right]$$

$$= \frac{1}{2\hbar} \left[ \hbar\ |\downarrow\downarrow\rangle + 0 + 0 + \hbar\ |\downarrow\downarrow\rangle \right]$$

$$= \frac{\cancel{2\hbar}}{\cancel{2\hbar}}\ |\downarrow\downarrow\rangle$$

$$\Rightarrow \quad |1\,{-1}\rangle = |\downarrow\downarrow\rangle$$

$\Rightarrow$ the states
$$|11\rangle = |\uparrow\uparrow\rangle$$
$$|10\rangle = \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$
$$|1-1\rangle = |\downarrow\downarrow\rangle$$
} triplet

are the three states with $s=1$ and $m=1,0,-1$

But we had 4 states!

clearly given $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ there are two linearly independent, orthonormal states

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \quad \text{and} \quad \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

Both have $S_z = 0$ but the first has $s=1$

Let's compute

$$\vec{S}^2 \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = (\vec{S}_1^2 + \vec{S}_2^2 + S_1^+ S_2^- + S_1^- S_2^+ + 2 S_1^z S_2^z)|\text{state}\rangle$$

but
$$\vec{S}_1^2 |\uparrow\downarrow\rangle = \tfrac{3}{4}\hbar^2 |\uparrow\downarrow\rangle \qquad \vec{S}_1^2 |\downarrow\uparrow\rangle = \tfrac{3}{4}\hbar^2 |\downarrow\uparrow\rangle$$
$$\vec{S}_2^2 |\uparrow\downarrow\rangle = \tfrac{3}{4}\hbar^2 |\uparrow\downarrow\rangle \qquad \vec{S}_2^2 |\downarrow\uparrow\rangle = \tfrac{3}{4}\hbar^2 |\downarrow\uparrow\rangle$$
$$S_1^+ S_2^- |\uparrow\downarrow\rangle = 0 \qquad S_1^+ S_1^- |\downarrow\uparrow\rangle = \hbar^2 |\uparrow\downarrow\rangle$$
$$S_1^- S_2^+ |\uparrow\downarrow\rangle = \hbar^2 |\downarrow\uparrow\rangle \qquad S_1^- S_1^+ |\downarrow\uparrow\rangle = 0$$
$$S_1^z S_2^z |\uparrow\downarrow\rangle = -\frac{\hbar^2}{4} |\uparrow\downarrow\rangle \qquad S_1^z S_2^z |\downarrow\uparrow\rangle = -\frac{\hbar^2}{4} |\downarrow\uparrow\rangle$$

$$\vec{S}^2 \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}\left(\frac{3}{4}\hbar^2 \times 2\right)|\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}\frac{3}{4}\hbar^2 \times 2\,|\downarrow\uparrow\rangle$$
$$+ \cdots = 0$$

$$\Rightarrow s = 0 \Rightarrow |0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \quad \underline{\underline{\text{Singlet}}}$$

$\Rightarrow$ given two $s = \frac{1}{2}$ we have ~~two~~ two states per spin and a total of 4 states : 3 states with $S=1$ and one state with $S=0$

Clearly $\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left[(\vec{S}_1+\vec{S}_2)^2 - \vec{S}_1^{\,2} - \vec{S}_2^{\,2}\right] =$

$$= \frac{1}{2}\left(\vec{S}^{\,2} - \vec{S}_1^{\,2} - \vec{S}_2^{\,2}\right) = \frac{\hbar^2}{2}\left(s(s+1) - \frac{3}{4} - \frac{3}{4}\right)$$

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{\hbar^2}{2}\left(s(s+1) - \frac{3}{2}\right)$$

$$\vec{S}_1 \cdot \vec{S}_2 \, |1,m\rangle = \frac{\hbar^2}{2}\left(1\cdot 2 - \frac{3}{2}\right)|1,m\rangle = \frac{\hbar^2}{4}\,|1,m\rangle \quad \text{triplet}$$

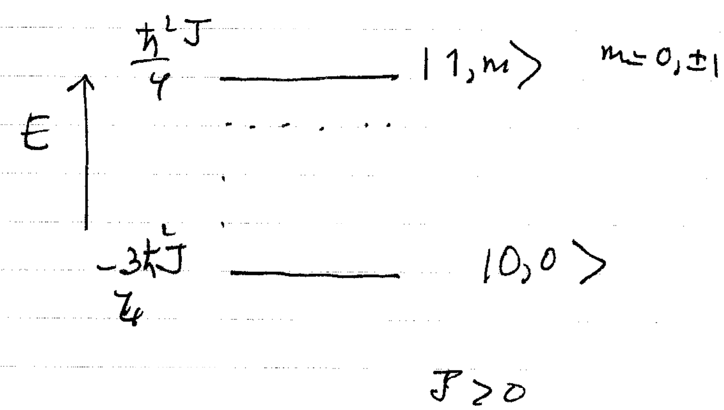
$$\vec{S}_1 \cdot \vec{S}_2 \, |0,0\rangle = \frac{\hbar^2}{2}\left(0 - \frac{3}{2}\right)|0,0\rangle = -\frac{3}{4}\hbar^2|0,0\rangle \quad \text{singlet}$$

Thus if $H = J\,\vec{S}_1 \cdot \vec{S}_2 = \frac{J}{2}\left(\vec{S}^{\,2} - \vec{S}_1^{\,2} - \vec{S}_2^{\,2}\right)$

Eigenstates are $|S, m\rangle$

$$H|1,m\rangle = \frac{J\hbar^2}{4}\,|1,m\rangle$$

$$H|0,0\rangle = -\frac{3J\hbar^2}{4}|0,0\rangle$$




Q: $P_0 = \frac{\hbar^2}{4} - \vec{S}_1 \cdot \vec{S}_2$

$P_0 |0,0\rangle = 0$

$P_1 = 1 - P_0 \quad , \quad P_1 |1,m\rangle = 0$

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## Addition of two angular momenta (general)

two ang. momenta $\vec{J}_1$, $\vec{J}_2$

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

$$\vec{J}_1^{\,2} \,|j_1, m_1\rangle = \hbar^2 j_1(j_1+1)\,|j_1 m_1\rangle \,, \qquad \vec{J}_2^{\,2}\,|j_2 m_2\rangle = \hbar^2 j_2(j_2+1)\,|j_2 m_2\rangle$$

$$J_1^z\,|j_1 m_1\rangle = \hbar m_1\,|j_1 m_1\rangle \,, \qquad J_2^z\,|j_2 m_2\rangle = \hbar m_2\,|j_2 m_2\rangle$$

$\Rightarrow$ Form the direct product

$$|j_1 m_1\rangle \times |j_2 m_2\rangle \equiv |j_1 j_2\, m_1 m_2\rangle$$

$$[\, j_1 \otimes j_2 \,]$$

Q: Can we construct the basis of eigenstates of $\vec{J}^2$ and $J_z$ ?

$$[J_i, J_j] = i\hbar\,\varepsilon_{ijk}\,J_k$$

$$\vec{J}^2 = \vec{J}_1^{\,2} + \vec{J}_2^{\,2} + 2\,\vec{J}_1\cdot\vec{J}_2$$

We want a basis $|j_1, j_2, j, m\rangle$ since $\vec{J}_1^{\,2}$, $\vec{J}_2^{\,2}$, $\vec{J}^2$, and $J_z$ commute with each other.

$$\vec{J}^2\,|j_1 j_2 j m\rangle = \hbar^2 j(j+1)\,|j_1 j_2 j m\rangle$$

$$J_z\,|j_1 j_2 j m\rangle = \hbar m\,|j_1 j_2 j m\rangle$$

$$\vec{J}_1^{\,2}\,|j_1 j_2 j m\rangle = \hbar^2 j_1(j_1+1)\,|j_1 j_2 j m\rangle$$

$$\vec{J}_2^{\,2}\,|j_1 j_2 j m\rangle = \hbar^2 j_2(j_2+1)\,|j_1 j_2 j m\rangle$$

# L18 Clebsch - Gordan Coefficients

We want to ~~write~~ find the linear combinations

$$|j_1 j_2 j m\rangle = \sum_{\substack{j_1' j_2' \\ m_1 m_2}} |j_1' j_2' m_1 m_2\rangle \underbrace{\langle j_1' j_2' m_1 m_2 | j_1 j_2 j m\rangle}_{\substack{\text{coefficients} \\ \equiv \\ \text{Clebsch-Gordan Coefficients}}}$$

Since

$$\langle j_1' j_2' m_1 m_2 | \vec{J}_1^{\,2} | j_1 j_2 j m\rangle = \hbar^2 j_1'(j_1'+1) \langle j_1' j_2' m_1 m_2 | j_1 j_2 j m\rangle$$

$$= \hbar^2 j_1 (j_1+1) \langle j_1' j_2' m_1 m_2 | j_1 j_2 j m\rangle$$

$\Rightarrow$ the coeff. is zero unless $\boxed{j_1 = j_1'}$ and $\boxed{j_2 = j_2'}$

Also

$$\langle j_1 j_2 m_1 m_2 | J_z | j_1 j_2 j m\rangle = (m_1 + m_2)\hbar \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$$

$$= m\hbar \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$$

$\Rightarrow$ $\boxed{m = m_1 + m_2}$ or the coeff. is zero.

$\Rightarrow$ the only $\neq 0$ coeffs. are of the form

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 j, m = m_1 + m_2\rangle$$

$$\Rightarrow |j_1 j_2 j m\rangle = \sum_{\substack{m_1, m_2 \\ m_1 + m_2 = m}} |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$$

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus \cdots \oplus |j_1 - j_2|$$

reps!

dimension $= (2j_1+1)(2j_2+1)$

irred. reps!

For $j_1, j_2$ fixed there are $\underline{2j_1+1}$ possible values of $\underline{m_1}$ and $\underline{2j_2+1}$ values of $\underline{m_2}$ $\Rightarrow$ there are $(2j_1+1)\times(2j_2+1)$ linearly independent states $|j_1 j_2\, j\, m\rangle$

But there is only $\underline{\text{one}}$ state with $j_1+j_2 = m_1+m_2$, there are $\underline{\text{two}}$ states with $m_1+m_2 = j_1+j_2-1$ (i.e. $m_1 = j_1,\ m_2 = j_2-1$ and $m_1 = j_1-1,\ m_2 = j_2$). In general there are as many diff. states with fixed $\underline{m}$ as we can choose $m_1$ and $m_2$ / $m_1+m_2 = m$.

For $m \geq |j_1-j_2|$ there are $j_1+j_2-m+1$ ~~different~~ ways while for $-|j_1-j_2| < m < |j_1-j_2|$ there are $j_1+j_2-|j_1-j_2|+1$ diff. ways and for $|m| < -|j_1-j_2|$ there are $j_1+j_2-|m|+1$ diff. ways.

e.g. $j_1 = 2,\ j_2 = 1$ we have 1 state with $m = \pm 3$, 2 states with $m = \pm 2$, 3 states with $m = \pm 1$ and 3 states with $m = 0$ with a total of $2\times 1 + 2\times 2 + 2\times 3 + 3 = 15 = (2\times 2+1)(2\times 1+1)$

What are the $\underline{\text{possible values of } j}$ that we can make with fixed $j_1$ and $j_2$?

We notice first that if we have $\underline{a}$ state $|j_1 j_2\, j\, m\rangle$ $\Rightarrow$ we can form all other states $|j_1 j_2\, j\, m'\rangle$ $(|m'| \leq j)$ by acting with $J_\pm = J_{1\pm} + J_{2\pm}$

$\Rightarrow$ we must break the $(2j_1+1)\times(2j_2+1)$ states into <u>multiplets</u> with given $j$.

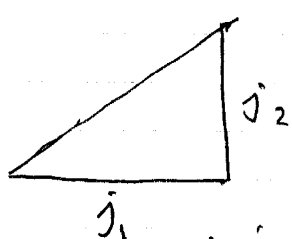
Since $m = j_1 + j_2$ is the largest $m$ $\Rightarrow j = j_1 + j_2$ is the largest $j$.

$\Rightarrow$ we must have a multiplet with $2j+1$ $(j = j_1 + j_2)$ states. What's next? The next largest value of $m$ is $j_1 + j_2 - 1$ (there are <u>two</u> such states but one has $j = j_1 + j_2$).

$\Rightarrow$ there are $2(j_1+j_2-1)+1$ states in that multiplet.

Next there is a state with highest $m = j_1 + j_2 - 2$ (there is a total of 3 of these states but two are already counted in the two others multiplets) $\Rightarrow \exists$ $2(j_1+j_2-2)+1$ ....

In general $|j_1 - j_2| \le j \le |j_1 + j_2|$



The possible values of $j$ must <u>differ</u> by <u>integers</u>

check $$\sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1) = (2j_1+1)(2j_2+1) \quad \checkmark$$

# Clebsch-Gordan Methods

## Orthogonality relations:

$$|j_1 j_2 j m\rangle = \sum_{\substack{m_1, m_2 \\ (m = m_1 + m_2)}} |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$$

$$\Rightarrow \quad \langle j_1 j_2 m_1' m_2' | j_1 j_2 m_1 m_2\rangle =$$

$$= \sum_{j, m} \langle j_1 j_2 m_1' m_2' | j_1 j_2 j m\rangle \langle j_1 j_2 j m | j_1 j_2 m_1 m_2\rangle$$

$$= \delta_{m_1 m_1'} \, \delta_{m_2 m_2'}$$

$$\Rightarrow \quad \sum_{jm} \langle j_1 j_2 m_1' m_2' | j_1 j_2 j m\rangle \langle j_1 j_2 j m | j_1 j_2 m_1 m_2\rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

The CGC's can be taken to be real

$$\Rightarrow \quad \langle j_1 j_2 j m | j_1 j_2 m_1 m_2\rangle = \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$$

and

$$\sum_{\substack{m_1 m_2 \\ (m_1 + m_2 = m)}} \langle j_1 j_2 j m | j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j' m'\rangle = \langle j_1 j_2 j m | j_1 j_2 j' m'\rangle$$

$$= \delta_{jj'} \delta_{mm'}$$

$\Rightarrow$ for $j = j'$, $m = m'$

$$\Rightarrow \quad 1 = \sum_{\substack{m_1, m_2 \\ (m_1 + m_2 = m)}} \langle j_1 j_2 j m | j_1 j_2 m_1 m_2\rangle^2$$

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## How to determine the CGC's

use the recursion relations from the m.e.

$$\langle j_1 j_2 m_1 m_2 | J_\pm | j_1 j_2 j m\rangle$$

acting on the right $\Rightarrow$ $\langle j_1 j_2 m_1 m_2 | J_\pm | j_1 j_2 j m\rangle =$

$$= \hbar\sqrt{j(j+1) - m(m\pm 1)}\ \langle j_1 j_2 m_1 m_2 | j_1 j_2\, j\, m\pm 1\rangle$$

acting on the left ($J_\pm = J_{1\pm} + J_{2\pm}$)

$$\langle j_1 j_2 m_1 m_2 | J_{1\pm} + J_{2\pm} | j_1 j_2 j m\rangle =$$

$$= \hbar\sqrt{j_1(j_1+1) - m_1(m_1\pm 1)}\ \langle j_1 j_2\, m_1\pm 1\, m_2 | j_1 j_2 j m\rangle$$

$$+ \hbar\sqrt{j_2(j_2+1) - m_2(m_2\pm 1)}\ \langle j_1 j_2\, m_1\, m_2\pm 1 | j_1 j_2 j m\rangle$$

$\Rightarrow$ recursion relations

$$\sqrt{j(j+1) - m(m\pm 1)}\ \langle j_1 j_2 m_1 m_2 | j_1 j_2\, j\, m\pm 1\rangle =$$

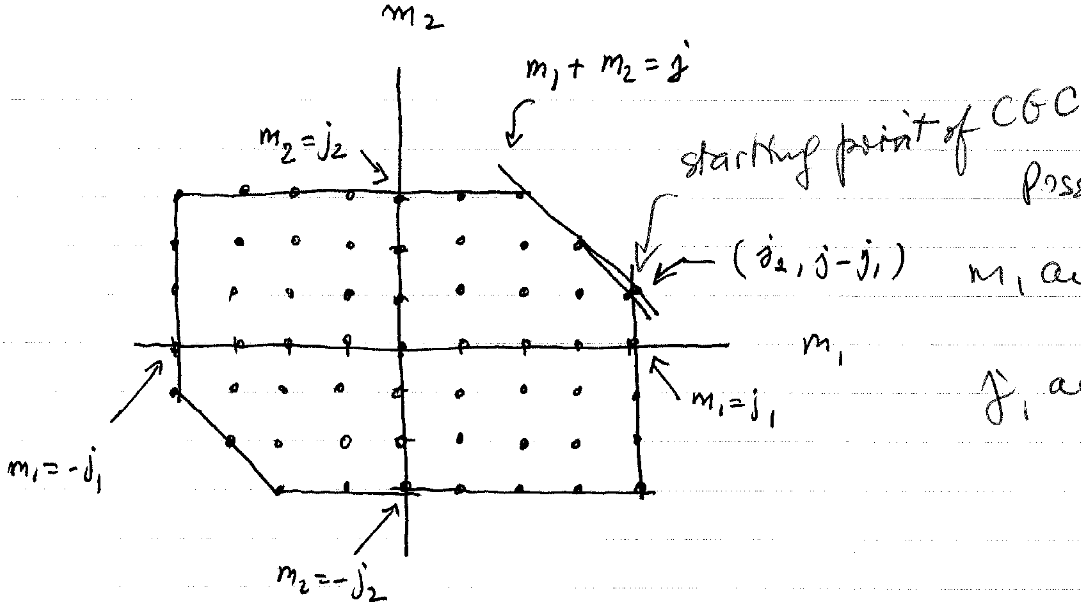
$$= \sqrt{j_1(j_1+1) - m_1(m_1\mp 1)}\ \langle j_1 j_2\, m_1\pm 1\, m_2 | j_1 j_2 j m\rangle$$

$$+ \sqrt{j_2(j_2+1) - m_2(m_2\mp 1)}\ \langle j_1 j_2\, m_1\, m_2\pm 1 | j_1 j_2 j m\rangle$$

We will use these rec. rel. + the condition

$$\sum_{\substack{m_1, m_2 \\ m_1+m_2=m}} |\langle j_1 j_2 j m | j_1 j_2 m_1 m_2\rangle|^2 = 1$$

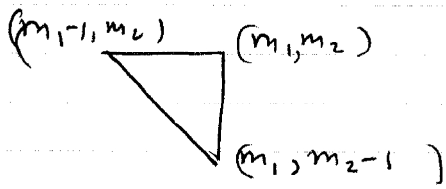
to determine the CGCs.




Possible values of $m_1$ and $m_2$ for fixed $j_1$ and $j_2$

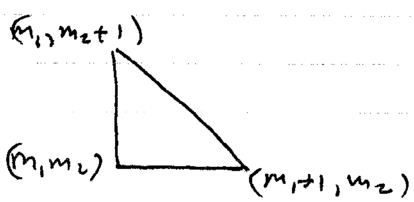
The recursion relations imply that if we know the coefficients $\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle$ for fixed $j$ and

$(m_1, m_2 - 1)$, $(m_1 - 1, m_2) \Rightarrow$ we know it for $(m_1, m_2)$




"upper" rec. rel.

and, for fixed $j$,




"lower" rec. rel.

In particular, if we set $m_1 = j_1$, we get (choosing the "upper" rec. rel.)

$$\sqrt{j(j+1) - m(m+1)}\ \langle j_1, j_2, j_1, m_2 | j_1, j_2, j, m+1 \rangle =$$

$$= \sqrt{j_2(j_2+1) - m_2(m_2+1)}\ \langle j_1, j_2, j_1, m_2 + 1 | j_1, j_2, j, m \rangle$$

Similarly ("lower")

$$\sqrt{j(j+1)-m(m-1)}\;\langle j_1, j_2, m_1, j_2 \mid j_1, j_2, j, m-1\rangle =$$

$$=\sqrt{j_1(j_1+1)-m_1(m_1+1)}\;\langle j_1, j_2, m_1+1, j_2 \mid j_1, j_2, j, m\rangle$$

We will now apply these relations to find the CGC's for the important case $j_1 = l \in \mathbb{Z}$ and $j_2 = \frac{1}{2}$

Clearly, we will only have $j = l+\frac{1}{2},\ l-\frac{1}{2}$.

The state $|j_1\, j_2\, m_1\, m_2\rangle = |l, \frac{1}{2}, l, \frac{1}{2}\rangle$ is the only state with $m = l+\frac{1}{2} \Rightarrow$ it must have $j = l+\frac{1}{2}$

$$\Rightarrow |l, \tfrac{1}{2}, l, \tfrac{1}{2}\rangle = |\underset{\substack{\uparrow\\ j_1}}{l},\ \underset{\substack{\uparrow\\ j_2}}{\tfrac{1}{2}},\ \underset{\substack{\uparrow\\ j}}{l+\tfrac{1}{2}},\ \underset{\substack{\uparrow\\ m}}{l+\tfrac{1}{2}}\rangle$$

$$\Rightarrow \quad \langle l, \tfrac{1}{2}, l, \tfrac{1}{2} \mid l, \tfrac{1}{2}, l+\tfrac{1}{2}, l+\tfrac{1}{2}\rangle = 1. \quad \text{(normalization)}$$

Also, for this case, the relations become

$$\sqrt{(l+\tfrac{1}{2})(l+\tfrac{3}{2})-m(m-1)}\;\langle l, \tfrac{1}{2}, m_1, \tfrac{1}{2} \mid l, \tfrac{1}{2}, l+\tfrac{1}{2}, m-1\rangle =$$

$$=\sqrt{l(l+1)-m_1(m_1+1)}\;\langle l, \tfrac{1}{2}, m_1+1, \tfrac{1}{2} \mid l, \tfrac{1}{2}; l+\tfrac{1}{2}, m\rangle$$

after substituting $m \to m+1$ and $m_1 = m+\frac{1}{2}$

we get

$$\sqrt{\left(\ell+\tfrac{1}{2}\right)\left(\ell+\tfrac{3}{2}\right)-m(m+1)}\;\left\langle \ell,\tfrac{1}{2},m_1-1,\tfrac{1}{2}\,\middle|\,\ell,\tfrac{1}{2},\ell+\tfrac{1}{2},m\right\rangle$$

$$=\sqrt{\ell(\ell+1)-m_1(m_1-1)}\;\left\langle \ell,\tfrac{1}{2},m_1,\tfrac{1}{2}\,\middle|\,\ell,\tfrac{1}{2},\ell+\tfrac{1}{2},m+1\right\rangle$$

Also $\quad j(j+1)-m(m\pm1)=(j\mp m)(j\pm m+1)$

$\Rightarrow$ (setting $m_1=m+\frac{1}{2}$ )

$$\left\langle \ell,\tfrac{1}{2},m-\tfrac{1}{2},\tfrac{1}{2}\,\middle|\,\ell,\tfrac{1}{2},\ell+\tfrac{1}{2},m\right\rangle=\sqrt{\frac{(\ell+m_1)(\ell-m_1+1)}{\left(\ell+\frac{1}{2}-m\right)\left(\ell+\frac{1}{2}+m+1\right)}}$$

$$\left\langle \ell,\tfrac{1}{2},m+\tfrac{1}{2},\tfrac{1}{2}\,\middle|\,\ell,\tfrac{1}{2},\ell+\tfrac{1}{2},m+1\right\rangle$$

$$=\sqrt{\frac{\ell+m+\frac{1}{2}}{\ell+m+\frac{3}{2}}}\;\left\langle \ell,\tfrac{1}{2},m+\tfrac{1}{2},\tfrac{1}{2}\,\middle|\,\ell,\tfrac{1}{2},\ell+\tfrac{1}{2},m+1\right\rangle$$

$\Rightarrow$ iterating this relation and using the normalization property

$$\Rightarrow \left\langle \ell,\tfrac{1}{2},m-\tfrac{1}{2},\tfrac{1}{2}\,\middle|\,\ell,\tfrac{1}{2},\ell+\tfrac{1}{2},m\right\rangle=\sqrt{\frac{\ell+m+\frac{1}{2}}{2\ell+1}}$$

Likewise,

| | $\langle j_1, m_1, \frac{1}{2}, m_2 \mid j_1, \frac{1}{2}, j, m \rangle$ | |
|---|---|---|
| $j$ \ $m$ | $m_2 = 1/2$ | $m_2 = -1/2$ |
| $j = j_1 + 1/2$ | $+\sqrt{\dfrac{j_1 + m + \frac{1}{2}}{2j_1 + 1}}$ | $+\sqrt{\dfrac{j_1 - m + \frac{1}{2}}{2j_1 + 1}}$ |
| $j = j_1 - 1/2$ | $-\sqrt{\dfrac{j_1 - m + \frac{1}{2}}{2j_1 + 1}}$ | $+\sqrt{\dfrac{j_1 + m + \frac{1}{2}}{2j_1 + 1}}$ |

## Applications to Spin-Orbit interactions

Using the CGC's we can construct the basis $|l\, s\, j\, m\rangle$ from the basis $|l\, m\, s\, m_s\rangle$. For $s = \frac{1}{2}$ and $l \in \mathbb{Z}$ the CGC's are given by the table above.

In the absence of spin it is standard to denote the angular momentum levels by letters as

$$\begin{array}{cccccc} l = & 0 & 1 & 2 & 3 & 4 \;\ldots \\ & s & p & d & e & f \;\ldots \end{array}$$

With spin it is common to use the modified notation

$${}^{2S+1}L_J$$

where $L = l$ (the orbital ang. momentum)
$J$ is $j$
$2S+1$ multiplicity due to spin
(for a single electron atom $2S+1 = 2$)

S P D E F

For multielectron atoms, $L$ is the <u>total</u> angular moment, $S$ is the <u>total</u> spin and $J$ is their ~~[illegible]~~ sum.

$\Rightarrow \quad {}^2P_{3/2} \qquad l=1, \; s=\frac{1}{2}, \quad j=3/2$

Gnd State of He: $\quad {}^1S_0 \qquad l=0, \; S=0, \; J=0$

Spin-Orbit effects are easily seen in this basis. Indeed

$$H_{SO} \sim \# \, \vec{L}\cdot\vec{S} = \# \, \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$$

$$\Rightarrow \vec{L}\cdot\vec{S}\,|l,s,j,m\rangle = \frac{\hbar^2}{2}\left(j(j+1) - l(l+1) - s(s+1)\right)|l,s,j,m\rangle$$

For $s=\frac{1}{2}$ and $j=l+\frac{1}{2}$

$$\Rightarrow \vec{L}\cdot\vec{S}\,|l,\tfrac{1}{2},l+\tfrac{1}{2},m\rangle = \frac{l\hbar^2}{2}|l,\tfrac{1}{2},l+\tfrac{1}{2},m\rangle$$

and, for $j=l-\frac{1}{2}$,

$$\vec{L}\cdot\vec{S}\,|l,\tfrac{1}{2},l-\tfrac{1}{2},m\rangle = -\left(\frac{l+1}{2}\right)\hbar^2|l,\tfrac{1}{2},l-\tfrac{1}{2},m\rangle$$

$\Rightarrow$ states with $\vec{L}$ antiparallel to $\vec{S}$ have lower energy.

The actual form of the S-O term in Hydrogen is

$$H_{SO} = \frac{e^2}{4m^2c^2r^3}\left(J^2 - L^2 - S^2\right)$$

$$\Rightarrow \langle l',\tfrac{1}{2},j',m'|H_{SO}|l,\tfrac{1}{2},j,m\rangle = \delta_{jj'}\delta_{mm'}\delta_{ll'} \times$$

$$\times \frac{e^2}{4m^2c^2}\langle n,l|\frac{1}{r^3}|n,l\rangle \hbar^2\left(j(j+1) - l(l+1) - \frac{3}{4}\right)$$

$j = l \pm \frac{1}{2}$ first order P.T. shift due to S.O.

$$E^{(1)}_{SO} = \frac{\hbar^2 e^2}{4m^2c^2} \left\langle \frac{1}{r^3} \right\rangle_{nl} \begin{bmatrix} l \\ -(l+1) \end{bmatrix} \begin{matrix} \leftarrow j = l + \frac{1}{2} \\ \leftarrow j = l - \frac{1}{2} \end{matrix}$$

Since $\left\langle \frac{1}{r^3} \right\rangle_{nl} = \frac{1}{a_0^3} \frac{1}{n^3 l(l+\frac{1}{2})(l+1)}$

$$\Rightarrow E^{(1)}_{SO} = \frac{1}{4} mc^2 \alpha^4 \frac{\begin{bmatrix} l \\ -(l+1) \end{bmatrix}}{n^3 l (l+\frac{1}{2})(l+1)} \qquad l \neq 0$$

For $l=0$, $\left\langle \frac{1}{r^3} \right\rangle \to \infty$ but for $l=0$ there is only one state, $j=\frac{1}{2}$ and $\langle \vec{L} \cdot \vec{S} \rangle = 0$ in that state. The result is that, for $l=0$, we get

$$E^{(1)}_{SO}(l=0) = \frac{1}{4} \frac{mc^2 \alpha^4}{n^3}$$ which is finite

(explanation in Dirac's theory!)

If one adds the relativistic correction to K.E., $H_T \approx -\frac{p^4}{8m^3c^2}$

we get $E^{(1)}_T = -\frac{1}{8m^3c^2} \langle nml | p^4 | nml \rangle$

$$= -\frac{1}{2} mc^2\alpha^4 \left[ -\frac{3}{4n^4} + \frac{1}{n^3(l+1/2)} \right]$$

$$\Rightarrow E^{(1)}_T + E^{(1)}_{SO} = -\frac{mc^2\alpha^2}{2n^2} \frac{\alpha^2}{n} \left[ \frac{1}{j+\frac{1}{2}} - \frac{3}{4n} \right] \qquad j = l \pm \frac{1}{2}$$

$\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137.\ldots}$ fine structure const.

Applications 2:

Scattering of $S=1/2$ particles with $S=0$ particles

Another example is the scattering of particles with ≠ spin. For simplicity we will consider a $S=1/2$ particle (such as a proton p) and a $S=0$ particle (such as a pion $\pi$). In this case the orbital angular momentum is the relative orbital angular momentum of the p and $\pi$ and $S=\frac{1}{2}$ is the spin of p.

When one does a scattering expt. the states are naturally described by $|l, s, m_l, m_s\rangle$.

The p-$\pi$ interactions are isotropic. This means that the Hamiltonian is invariant under rotations ⇒ the total angular momentum $\vec{J}$ is conserved.

$$\vec{J} = \vec{L} + \vec{S} \quad \Rightarrow \quad [H, \vec{J}] = 0$$

but $L_z$ and $S_z$ are not conserved. (i.e. spin flip)

⇒ the quantum #'s, s j m are not expected to change although l might (we will see that this is not the case).

At the level of the Born-approx., the scattering amplitude is

$$\sim \langle l' s' j' m' | H_{int} | l s j m \rangle$$

since $[H_{int}, J_z] = 0$

$$\Rightarrow \langle l's'j'm'|[H_{int}, J_z]|lsjm\rangle = 0$$

But

$$\langle l's'j'm'|[H_{int}, J_z]|lsjm\rangle = (m-m')\langle l's'j'm'|H_{int}|lsjm\rangle$$

$$\Rightarrow \quad \langle l's'j'm'|H_{int}|lsjm\rangle = 0 \quad \underline{\text{unless}} \quad \underline{m = m'}$$

The same line of argument shows that since

$[H_{int}, \vec{J}^2] = 0 \quad \Rightarrow \quad \underline{j = j'}$

The allowed values of $j$ are

$j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$

$\Rightarrow$ given $j$, the allowed values of $l$ are $j \pm \frac{1}{2}$

since $l' = j \pm \frac{1}{2} \Rightarrow \underset{\text{final}}{\underset{\uparrow}{l'}} = \underset{\text{initial}}{\underset{\uparrow}{l}}, l \pm 1$

$H_{int}$ is invariant under $\underline{\text{Parity}}$, $[H_{int}, P] = 0$

and the parity of a state of angular momentum $l$ is $(-1)^l$ (spin is invariant)

$\Rightarrow (-1)^l = (-1)^{l'} \Rightarrow l - l'$ must be $\underline{\text{even}}$

$\Rightarrow \underline{l' = l}$

Furthermore, rotational invariance $\Rightarrow$

$\langle lsjm|H_{int}|lsjm\rangle \equiv B(l, j)$ (not of $\underline{m}$)

The scattering amplitude (in the Born app.) involves the matrix element $\cancel{\langle l's'j'm'|H_i}$ $\langle l' s\, m_l' m_s' | H_{int} | l s\, m_l m_s\rangle$

Changing basis

$$\langle l' s\, m_l' m_s' | H_{int} | l s\, m_l m_s\rangle =$$

$$= \sum_{\substack{j, j' \\ m, m'}} \langle l' s\, m_l' m_s' | l' s\, j' m'\rangle \langle l' s\, j' m' | H | l s\, j m\rangle \langle l s j m | l s\, m_l m_s\rangle$$

$$\equiv \delta_{l l'}\, \delta_{m_l + m_s,\, m_l' + m_s'} \sum_{j = l \pm \frac{1}{2}} \overbrace{\phantom{x}}^{B(l,j)} \langle l s\, m_l' m_s' | l s j m\rangle \langle l s j m | l s\, m_l m_s\rangle$$

$\Rightarrow$ we only need to know the amplitudes $B(l, l \pm \frac{1}{2})$ and the CGC's

Fall 2002 $\rightarrow$ Skip to ~~Interaction of Radiation with matter~~ Spin and Identical Particles

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## Rotations and Tensor Operators

If $\vec{J}$ is the total angular momentum of a physical system, the operator

$$R[\alpha] = e^{-\frac{i}{\hbar}\vec{\alpha}\cdot\vec{J}}$$

acting on the right, ~~rotates the states~~ rotates the states in a (+) sense about the axis $\frac{\vec{\alpha}}{|\vec{\alpha}|}$ by an angle $|\vec{\alpha}|$. The states $|jm\rangle$ are eigenstates of $\vec{J}^2$ and $J_z$. The state

$R[\alpha]|jm\rangle$ remains an eigenstate of $\vec{J}^2$ (with the same e.v.), i.e. a uniform rotation does not change the angular momentum of the system.

$$\Rightarrow \quad [R[\alpha], \vec{J}^2] = 0 \Rightarrow \quad \vec{J}^2 R[\alpha]|jm\rangle = R[\alpha]\vec{J}^2|jm\rangle = \hbar^2 j(j+1) R[\alpha]|jm\rangle$$

However $R[\alpha]|jm\rangle$ is not an eigenstate of $J_z$ with e.v. $\hbar m$.

$\Rightarrow$ Since the set $\{|jm\rangle\}$ is complete

$$\Rightarrow \quad R[\alpha]|jm\rangle = \sum_{m''=-j}^{j} d^{(j)}_{m''m}[\alpha]\,|jm''\rangle$$

(since $j$ does not change)

$$\Rightarrow \quad d^{j}_{m'm}[\alpha] = \langle jm'|\, e^{-\frac{i}{\hbar}\vec{\alpha}\cdot\vec{J}}\,|jm\rangle$$

which is a $2j+1 \times 2j+1$ matrix.

Since $R[\gamma] = R[\beta]R[\alpha] \Rightarrow d^{(j)}_{m'm}(\gamma) = \sum_{m''} d^{(j)}_{m'm''}(\beta)\, d^{(j)}_{m''m}(\alpha)$

$d^{(j)}[\gamma] = d^{(j)}[\beta]\, d^{(j)}[\alpha]$ , the $d$'s are a representation of the rotation group.

properties:

① $d^{(j)}_{mm'}(0) = \delta_{mm'}$ (identity)

② $\left[d^{(j)}(\alpha)^{\dagger}\right]_{m'm} = d^{(j)}_{mm'}(\alpha)^{*} = \langle jm'|\, e^{-\frac{i}{\hbar}\vec{J}\cdot\vec{\alpha}}\,|jm\rangle^{*}$

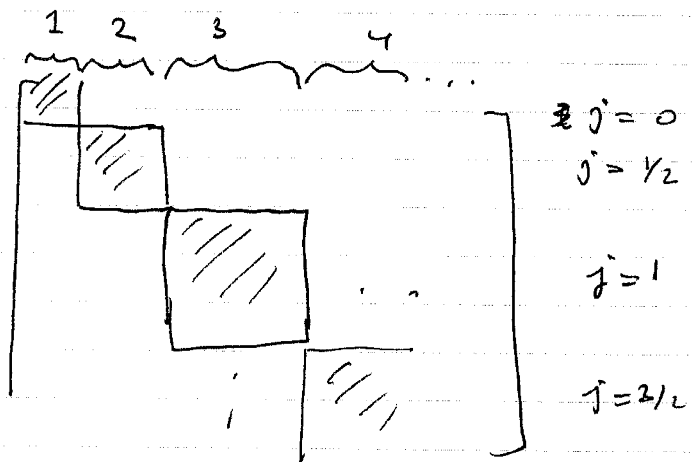
$= \langle jm'|\, e^{\frac{i}{\hbar}\vec{J}\cdot\vec{\alpha}}\,|jm\rangle$

$= d^{(j)}_{m'm}(-\alpha)$

$$d^{(j)}(\alpha)^{\dagger} = d^{(j)}(-\alpha)$$

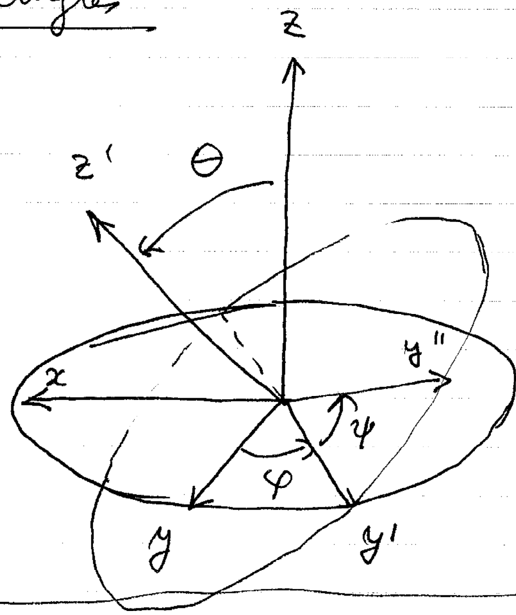
$$d^{(j)}(\alpha)^{\dagger}\, d^{(j)}(\alpha) = I$$

③ For $j$ fixed, the set $\{|jm\rangle\}$ is <u>irreducible</u> in the sense that it is the <u>smallest</u> set of states (with the same $j$) that transform into each other under the <u>action</u> of the rotation group. Conversely, for general but not fixed $j$, the $d$-matrices can be reduced to the block-diagonal form

$d =$ 


Thus, these matrices ~~$|jm\rangle$~~ $d^{(j)}_{mm'}$ with $j = \frac{1}{2}, 1$ are reducible but the matrices $d^{j}_{mm'}$ for $j = 1/2$ and $d^{j}_{mm'}$ for $j = 1$ are <u>irreducible</u>.

## Euler Angles




① rotate about $\hat{z}$ by $\varphi$
$(\hat{y} \to \hat{y}')$

② rotate about $\hat{y}'$ by ~~an~~ angle $\theta$ $(\hat{z} \to \hat{z}')$

③ rotate about $\hat{z}'$ by an angle $\psi$
$(\hat{y}' \to \hat{y}'')$

$$\boxed{R[\varphi,\theta,\psi] = e^{-\frac{i}{\hbar}\psi J_{z'}}\, e^{-\frac{i}{\hbar}\theta J_{y'}}\, e^{-\frac{i}{\hbar}\varphi J_z}}$$

But

$$J_{y'} = e^{-\frac{i}{\hbar}\varphi J_z}\; J_y \; e^{\frac{i}{\hbar}\varphi J_z}$$

$$\Rightarrow \quad f(J_{y'}) = e^{-\frac{i}{\hbar}\varphi J_z} f(J_y)\, e^{\frac{i}{\hbar}\varphi J_z}$$ (expand!)

$$\Rightarrow \quad e^{-\frac{i\theta}{\hbar} J_{y'}} = e^{-\frac{i}{\hbar}\varphi J_z}\, e^{-\frac{i}{\hbar}\theta J_y}\, e^{\frac{i}{\hbar}\varphi J_z}$$

Similarly

$$e^{-\frac{i}{\hbar}\psi J_{z'}} = e^{-\frac{i}{\hbar}\theta J_{y'}}\, e^{-\frac{i}{\hbar}\psi J_z}\, e^{\frac{i}{\hbar}\theta J_{y'}}$$

$$\Rightarrow \quad R[\varphi,\theta,\psi] = e^{-\frac{i}{\hbar}\varphi J_z}\, e^{-\frac{i}{\hbar}\theta J_y}\, \cancel{e^{\frac{i}{\hbar}\varphi J_z}}\, e^{-\frac{i}{\hbar}\psi J_z}\, \cancel{e^{-\frac{i}{\hbar}\varphi J_z}}\, \cancel{e^{\frac{i}{\hbar}\theta J_y}}\, \cancel{e^{\frac{i}{\hbar}\varphi J_z}}$$

$$\cancel{e^{-\frac{i}{\hbar}\varphi J_z}}\, \cancel{e^{-\frac{i}{\hbar}\theta J_y}}\, \cancel{e^{\frac{i}{\hbar}\varphi J_z}}\, \cancel{e^{-\frac{i}{\hbar}\varphi J_z}}$$

$$\Rightarrow \boxed{R[\varphi,\theta,\psi] = e^{-\frac{i}{\hbar}\varphi J_z}\, e^{-\frac{i}{\hbar}\theta J_y}\, e^{-\frac{i}{\hbar}\psi J_z}}$$

(note the unprimed indices!)

$$\Rightarrow \quad d^{(j)}_{mm'}(\varphi,\theta,\psi) = \langle jm|\, e^{-\frac{i}{\hbar}\varphi J_z}\, e^{-\frac{i}{\hbar}\theta J_y}\, e^{-\frac{i}{\hbar}\psi J_z}\,|jm'\rangle$$

$$= e^{\frac{-im\varphi}{\cancel{\hbar}}}\, e^{-im'\psi}\, \langle jm|\, e^{-\frac{i}{\hbar}\theta J_y}\,|jm'\rangle$$

$$\Rightarrow \quad d^{(j)}_{mm'}(\varphi,\theta,\psi) = e^{-im\varphi}\, e^{-im'\psi}\, d^{(j)}_{mm'}(\theta)$$

$\Rightarrow$ we only need the matrix for rotations about $\hat{y}$ by an angle $\theta$.

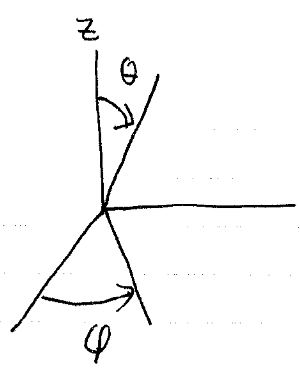
It turns out (see Baym page 386) that $d^{(j)}_{mm'}(\theta)$ is real. and that it satisfies

$$d^{(j)}_{-m,-m'}(\theta) = (-1)^{2j-m-m'}\, d^{(j)}_{mm'}(\theta)$$

Let's show that

$$d^{(j)}_{m,0}(\theta,\varphi) = \sqrt{\frac{4\pi}{2j+1}}\; Y^*_{jm}(\theta,\varphi)$$

Note that $R_{\varphi,\theta,0}$ is the rotation that brings the z axis to $(\varphi,\theta)$ (in spherical coords.)




$$\Rightarrow \quad |\theta=0,\varphi=0\rangle \rightarrow |\theta,\varphi\rangle = R_{\varphi,\theta,0}\,|\theta=0,\varphi=0\rangle$$

$$\Rightarrow \quad \sum_{m'} \langle jm|\, R_{\varphi,\theta,0}\,|jm'\rangle\langle jm'|\theta=0,\varphi=0\rangle =$$

$$= \langle jm|\theta,\varphi\rangle$$

$$\Rightarrow \quad \sum_{m'} d^{(j)}_{mm'}(\varphi,\theta,0)\; Y^*_{jm'}(0,0) = Y^*_{jm}(\theta,\varphi)$$

But $$Y_{jm'}(\theta,0) = \delta_{m',0}\sqrt{\frac{2j+1}{4\pi}}$$

$$\Rightarrow \quad d^{(j)}_{m0}(\varphi,\theta,0) = \sqrt{\frac{4\pi}{2j+1}}\; Y^*_{jm}(\theta,\varphi) \equiv d^{(j)}_{m0}(\varphi,\theta,\psi)$$

(since it's indep. of $\psi$)

Furthermore from

$$R[\alpha]\,|jm\rangle = \sum_{m''=-j}^{j} |jm''\rangle\, d^{(j)}_{m''m}[\alpha]$$

we get

$$\langle\theta,\varphi|\,R[\alpha]\,|jm\rangle = \langle\theta',\varphi'|\,jm\rangle = Y_{jm}(\theta',\varphi')$$

where $(\theta',\varphi')$ is the direction $(\theta,\varphi)$ rotated by $-|\vec{\alpha}|$ about $\frac{\vec{\alpha}}{|\alpha|}$.

$$\Rightarrow \quad Y_{jm}(\theta',\varphi') = \sum_{m'=-j}^{j} Y_{jm'}(\theta,\varphi)\, d^{(j)}_{m'm}[\alpha]$$

$$\Rightarrow \quad Y_{j0}(\theta',0) = \sqrt{\frac{4\pi}{2j+1}}\sum_{m=-j}^{j} Y_{jm}(\theta,\varphi)\, Y^*_{jm}(\theta_\alpha,\varphi_\alpha)$$

Euler angles of $[\alpha]$.

Since $\theta'$ is the angle between the directions with spherical coords $(\theta,\varphi)$ and $(\theta_\alpha,\varphi_\alpha) \Rightarrow$ addition thm.

Consider now the eigenstates $|j_1 j_2 m_1 m_2\rangle$, for fixed $j_1, j_2$ of two angular momenta $\vec{J}_1$ and $\vec{J}_2$. To perform rotations on these states we act with the operator $e^{-\frac{i}{\hbar}\vec{\alpha}\cdot\vec{J}}$, where $\vec{J} = \vec{J}_1 + \vec{J}_2$.

But the rotation ops. do not act irreducibly on the states $|j_1 j_2 m_1 m_2\rangle$ since, as we know, we can write the $|j_1 j_2 m_1 m_2\rangle$ states as linear combinations of the states $|j_1 j_2 j m\rangle$ which are eigenstates of $\vec{J}^2$ and $J_z$ and, under rotations, states with $\neq j$ do not mix.

$\Rightarrow$ the set of $(2j_1+1)\times(2j_2+1)$ $\underbrace{\text{states}}_{|j_1 j_2 m_1 m_2\rangle}$ must break up into groups of states for the possible values of $j$ and the states with a given $j$ transform only into each other under rotations.

How does $|j_1 j_2 m_1 m_2\rangle$ transform?

Since $e^{-\frac{i}{\hbar}\vec{\alpha}\cdot\vec{J}} = e^{-\frac{i}{\hbar}\vec{\alpha}\cdot\vec{J}_1}\, e^{-\frac{i}{\hbar}\vec{\alpha}\cdot\vec{J}_2}$

$$\Rightarrow e^{-\frac{i}{\hbar}\vec{\alpha}\cdot\vec{J}}\,|j_1 j_2 m_1 m_2\rangle = \sum_{m_1', m_2'} |j_1 j_2 m_1' m_2'\rangle\, d^{(j_1)}_{m_1 m_1'}[\alpha]\, d^{(j_2)}_{m_2 m_2'}[\alpha]$$

$$\Rightarrow \langle j_1 j_2 m_1' m_2' |\, e^{-\frac{i}{\hbar}\vec{\alpha}\cdot\vec{J}} \,| j_1 j_2 m_1 m_2\rangle =$$

$$= d^{(j_1)}_{m_1' m_1}[\alpha] \quad d^{(j_2)}_{m_2' m_2}[\alpha]$$

This is the matrix element of a $(2j_1+1)(2j_2+1) \times (2j_1+1)(2j_2+1)$ dimensional matrix

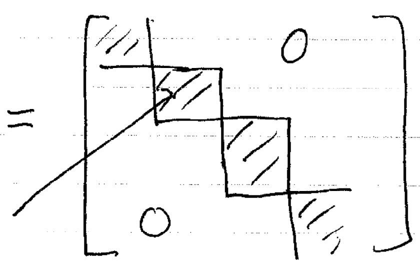
$$d^{(j_1)} \otimes d^{(j_2)}$$

the <u>direct</u> product of the two matrices $d^{(j_1)}$ and $d^{(j_2)}$

$$\Rightarrow \left[d^{(j_1)} \otimes d^{(j_2)}\right]_{m_1' m_2', m_1 m_2} = d^{(j_1)}_{m_1' m_1} \; d^{(j_2)}_{m_2' m_2}$$

This set of (direct product) matrices is <u>reducible</u> since in the basis $|j_1 j_2 j m\rangle$ they take a block diagonal form

$$d^{(j_1)} \otimes d^{(j_2)} =$$



each submatrix for a fixed $j$

The change of basis is given by the CGC's

$$\langle j_1 j_2 m_1' m_2' | e^{-\frac{i}{\hbar}\vec{\alpha}\cdot\vec{J}} | j_1 j_2 m_1 m_2 \rangle$$

$$= \sum_{j m m'} \langle j_1 j_2 m_1' m_2' | j_1 j_2 j m' \rangle \langle j_1 j_2 j m' | e^{-\frac{i}{\hbar}\vec{\alpha}\cdot\vec{J}} | j_1 j_2 j m \rangle \langle j_1 j_2 j m | j_1 j_2 m_1 m_2 \rangle$$

but $\langle j_1 j_2 j m' | e^{-\frac{i}{\hbar}\vec{\alpha}\cdot\vec{J}} | j_1 j_2 j m \rangle = d^{(j)}_{m'm}[\alpha]$

$$\Rightarrow \boxed{d^{(j_1)}_{m_1' m_1}[\alpha]\, d^{(j_2)}_{m_2' m_2}[\alpha] = \sum_{j} \sum_{m, m'} \langle j_1 j_2 m_1' m_2' | j_1 j_2 j m' \rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle\, d^{(j)}_{m'm}[\alpha]}$$

or, using orthogonality of the CGC's,

$$d^{(j)}_{m'm}[\alpha]\, \delta_{jj'} = \sum_{m_1 m_1'} \sum_{m_2 m_2'} \langle j_1 j_2 m_1' m_2' | j_1 j_2 j m' \rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle \times d^{(j_1)}_{m_1' m_1}[\alpha]\, d^{(j_2)}_{m_2' m_2}[\alpha]$$

e.g. if $j_1, j_2$ are integers

$\Rightarrow$

$$Y_{j_1 m_1}(\theta,\varphi)\, Y_{j_2 m_2}(\theta,\varphi) = \sum_{jm} \sqrt{\frac{(2j_1+1)(2j_2+1)}{4\pi\,(2j+1)}} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle \langle j_1 j_2 0 0 | j_1 j_2 j 0 \rangle\, Y_{jm}(\theta,\varphi)$$

$\Rightarrow$ products of $Y_{jm}$'s are l.c.'s of $Y_{jm}$'s

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## Tensor operators and the Wigner – Eckart Thm.

The states $\{|jm\rangle\}$ transform into each other under rotations following the transformation law

$$R[\alpha]\,|jm\rangle = \sum_{m''=-j}^{j} |jm''\rangle\, d^{(j)}_{m''m}[\alpha]$$

How do the operators themselves transform?

An arbitrary operator in this space has the form

$$A = \sum_{mm'} |jm\rangle\langle jm'|\; A^{(j)}_{mm'}$$

$\Rightarrow \quad A \rightarrow A_\alpha = R[\alpha]\; A\; R[\alpha]^{-1}$ is the transformation law.

$$\Rightarrow \quad \langle \phi | A | \psi \rangle = \langle \phi_\alpha | R[\alpha]\, A\, R[\alpha]^{-1} | \psi_\alpha \rangle$$

$$= \langle \phi_\alpha | A_\alpha | \psi_\alpha \rangle$$

i.e. the m.e. of $A$ are equal to the m.e. of the rotated op. $A_\alpha \equiv R[\alpha]\, A\, R[\alpha]^{-1}$ in the rotated states $|\psi_\alpha\rangle = R[\alpha]|\psi\rangle$, $|\phi_\alpha\rangle = R[\alpha]|\phi\rangle$

However some operators are scalars under rotations; i.e. they are invariant. Thus $\boxed{\text{if } [A, \vec{J}] = 0}$

$\Rightarrow \quad \boxed{[A, R[\alpha]] = 0} \quad \Rightarrow \quad A_\alpha = R[\alpha]\, A\, R[\alpha]^{-1} = R[\alpha] R[\alpha]^{-1} A = A$

$\Rightarrow \quad \boxed{A_\alpha = A}$ (scalar)

In general, ~~arbitrary~~ arbitrary ops. are not scalars but transform under rotations. Most ops. do not have simple transf. laws. There exists a set of ops., called tensor ops., that transform simply under rotations (~~these~~ ops. are a basis in the space of operators).

Let us define the irreducible tensor operators $T^{(k)}$ (of order k) as the set of $2k+1$ operators $T_q^{(k)}$, with $-k \leq q \leq k$ ~~[illegible]~~, which transform as follows

$$\boxed{R[\alpha]\, T_q^{(k)}\, R[\alpha]^{-1} = \sum_{q'=-k}^{k} T_{q'}^{(k)}\, d_{q'q}^{(k)}[\alpha]}$$

(i.e. they have the same transf. law as the states) where $d_{q'q}^{(k)}[\alpha]$ are the matrices ~~of an~~ m.e. of the irreducible representation of the rot. group (of dim. $2k+1$) ~~of dimension 2k+1~~.

Clearly

1. $T^{(0)}$ is a scalar (dimension 1)
2. $T^{(1)}$ is a vector (dimension 3)
3. $T^{(1/2)}$ is a spinor (dimension 2)

$\Rightarrow$ the irreducibility of $d^{(k)} \Rightarrow$ irreducibility of $T^{(k)}$.

For infinitesimal rotations

$$R[\alpha] = e^{-\frac{i}{\hbar}\vec{\alpha}\cdot\vec{J}} \approx I - \frac{i}{\hbar}\vec{\alpha}\cdot\vec{J} + \cdots$$

$$\Rightarrow \quad R[\alpha]\, T_q^{(k)}\, R[\alpha]^{-1} \approx \left(I - \frac{i}{\hbar}\vec{\alpha}\cdot\vec{J} + \cdots\right)\left(T_q^{(k)}\right)\left(I + \frac{i}{\hbar}\vec{\alpha}\cdot\vec{J} + \cdots\right)$$

$$= T_q^{(k)} - \frac{i}{\hbar}\left[\vec{\alpha}\cdot\vec{J},\, T_q^{(k)}\right] + \cdots$$

$$d^{(k)}_{q'q}[\alpha] = \langle kq'|\, e^{-\frac{i}{\hbar}\vec{\alpha}\cdot\vec{J}}\, |kq\rangle \approx \delta_{q'q} - \frac{i}{\hbar}\vec{\alpha}\cdot\langle kq'|\vec{J}|kq\rangle + \cdots$$

$$\Rightarrow \quad R[\alpha]\, T_q^{(k)}\, R[\alpha]^{-1} \approx T_q^{(k)} - \frac{i}{\hbar}\left[\vec{\alpha}\cdot\vec{J},\, T_q^{(k)}\right] + \cdots$$

$$= \sum_{q'=-k}^{k} T_{q'}^{(k)}\left[\delta_{q'q} - \frac{i}{\hbar}\vec{\alpha}\cdot\langle kq'|\vec{J}|kq\rangle + \cdots\right]$$

$$= T_q^{(k)} - \frac{i}{\hbar}\vec{\alpha}\cdot\sum_{q'=-k}^{k}\langle kq'|\vec{J}|kq\rangle + \cdots$$

$$\Rightarrow \quad \boxed{\left[\vec{\alpha}\cdot\vec{J},\, T_q^{(k)}\right] = \sum_{q'=-k}^{k} T_{q'}^{(k)}\langle kq'|\,\vec{\alpha}\cdot\vec{J}\,|kq\rangle \quad \cancel{T_q^{(k)}}}$$

$$\Rightarrow \quad \boxed{\left[J_z,\, T_q^{(k)}\right] = \sum_{q'=-k}^{k} \cancel{\langle kq'|}\; T_{q'}^{(k)}\, \langle kq'|\, J_z\, |kq\rangle}$$

$$\Rightarrow \quad \boxed{\left[J_z,\, T_q^{(k)}\right] = \hbar q\, T_q^{(k)}}$$

$$[J_\pm, T_q^{(k)}] = \sum_{q'=-k}^{k} T_{q'}^{(k)} \langle k q' | J_\pm | k q \rangle$$

$$\Rightarrow \quad [J_\pm, T_q^{(k)}] = \hbar \, T_{q\pm 1}^{(k)} \sqrt{k(k+1) - q(q\pm 1)}$$

These eqns. are fully equivalent to the transf. law for finite rotations but much easier to deal with.

Example

a vector operator $\vec{V}$ obeys the commutation laws

$$[J_i, V_j] = i\hbar \, \varepsilon_{ijk} V_k \qquad (i,j,k = x,y,z)$$

$\Rightarrow$ it must be equivalent to the laws derived above specialized for vector operators ($k=1$)

Since $[J_z, V_z] = 0 \Rightarrow V_{q=0}$ must be $V_z$

$$V_{q=0} = V_z$$

we find the identifications:

$$V_{q=\pm 1} = \mp \frac{V_x \pm i V_y}{\sqrt{2}}$$

let $\vec{V} = \vec{r}$

$$\Rightarrow \quad r_0 = z \quad , \quad r_1 = -\frac{x+iy}{\sqrt{2}} \quad , \quad r_{-1} = \frac{x-iy}{\sqrt{2}}$$

$$\left.\begin{aligned} x &= r\cos\varphi \sin\theta \\ y &= r\sin\varphi \sin\theta \\ z &= r\cos\theta \end{aligned}\right\} \Rightarrow \quad \begin{aligned} r_1 &= -\frac{r}{2} \sin\theta \, e^{i\varphi} \\ r_0 &= r\cos\theta \\ r_{-1} &= \frac{r}{2} \sin\theta \, e^{-i\varphi} \end{aligned}$$

spherical coords.

$$\Rightarrow \quad r_q = \sqrt{\frac{4\pi}{3}}\ r \quad Y_{1q}(\theta,\varphi) \equiv \sqrt{\frac{4\pi}{3}}\ P_{1,q}$$

and $P_{1,q}$ is a first order homogeneous polynomial.

For an arbitrary vector operator $\vec{V}$ we have

$$V_q = \sqrt{\frac{4\pi}{3}}\ P_{1,q}(V_x, V_y, V_z)$$

It can be shown (see Baym p. 370-371) that $(l, m \in \mathbb{Z})$

$P_{lm}(x,y,z) = r^l\ Y_{lm}(\theta,\varphi)$, which is a homogeneous polynomial of order $l$, has the right transformation laws, i.e.

$$R[\alpha]\ P_{lm}\ R^{-1}[\alpha] = \sum_{m'} P_{lm'}\ d^{(l)}_{m'm}[\alpha]$$

<u>Other properties of Tensor Ops.</u>

Let $|\lambda, jm\rangle$ be a state ($\lambda$ are other quantum #'s)

$$\Rightarrow \quad T^{(k)}_q |\lambda jm\rangle \text{ is a state with } J_z = \hbar(m+q)$$

$$\Leftarrow \quad J_z T^{(k)}_q |\lambda jm\rangle = [J_z, T^{(k)}_q]|\lambda jm\rangle + T^{(k)}_q J_z |\lambda jm\rangle$$

$$= \hbar q\ T^{(k)}_q |\lambda jm\rangle + \hbar m\ T^{(k)}_q |\lambda jm\rangle$$

$$\Rightarrow \quad J_z T^{(k)}_q |\lambda jm\rangle = \hbar(q+m)\ T^{(k)}_q |\lambda jm\rangle$$

It is also easy to show that

$$R[\alpha]\ T^{(k)}_q\ |\lambda jm\rangle = e^{-i(q+m)\varphi}\ T^{(k)}_q |\lambda jm\rangle$$

The operators $T_q^{(k)}$ are not eigenstates of $\vec{J}^2$ since acting on a state $|j_1 m_1\rangle$ with $T_q^{(k)}$ is like adding two angular momenta with e.v.'s $kq$ and $j_1 m_1$. The state

$$|\lambda j m\rangle = \sum_{q, m_1} T_q^{(k)} |\lambda j_1 m_1\rangle \underbrace{\langle k j_1 q m_1 | k j_1 j m\rangle}_{CGC}$$

is an eigenstate of $\vec{J}^2$ with e.v. $\hbar^2 j(j+1)$

To show this we compute $R[\alpha]|jm\rangle$

$$R[\alpha]|\lambda j m\rangle = \sum_{q, m_1} R[\alpha]\, T_q^{(k)} |\lambda j_1 m_1\rangle \langle k j_1 q m_1 | k j_1 j m\rangle$$

$$= \sum_{q, m_1} \left(R[\alpha]\, T_q^{(k)} R[\alpha]^{-1}\right) R[\alpha] |\lambda j_1 m_1\rangle \langle k j_1 q m_1 | k j_1 j m\rangle$$

$$= \sum_{q, m_1} \sum_{q'} T_{q'}^{(k)} d_{q'q}^{(k)}[\alpha] \sum_{m_1'} |\lambda j_1 m_1'\rangle d^{(j_1)}_{m_1' m_1}[\alpha] \langle k j_1 q m_1 | k j_1 j m\rangle$$

$$= \sum_{q' m_1'} T_{q'}^{(k)} |\lambda j_1 m_1'\rangle \sum_{q\, m_1} d_{q'q}^{(k)}[\alpha]\, d^{(j_1)}_{m'm}[\alpha] \langle k j_1 q m_1 | k j_1 j m\rangle$$

using the orthogonality property of the CGC's ~~the fixed~~ and the decomposition of the $d \otimes d$ we get

$$R[\alpha]|\lambda j m\rangle = \sum_{q' m_1' m'} T_{q'}^{(k)} |\lambda j_1 m_1'\rangle \langle k j_1 q' m_1' | k j_1 j m'\rangle\, d^{(j)}_{m'm}[\alpha]$$

$$= \sum_{m'} |\lambda j m'\rangle\, d^{j}_{m'm}[\alpha]$$

$\Rightarrow |\lambda j m\rangle$ transforms like a state with ev's $j, m$.

Averages (or integrals) over rotations:

$$\omega = (\varphi, \theta, \psi)$$

- $\int d\omega \, f(\omega) \equiv \underbrace{\frac{1}{2}\int_{-1}^{1} d\cos\theta \int_0^{4\pi} \frac{d\varphi}{4\pi} \int_0^{4\pi} \frac{d\psi}{4\pi}}_{\text{invariant (Haar) measure}} f(\varphi, \theta, \psi)$

($4\pi$'s due to spin)

- $\boxed{\int d\omega = 1}$ $\qquad \int d(R[\alpha]\omega) = \int d\omega$ (invariance)

- $\int d\omega \, d^{(j)}_{mm'}(\varphi, \theta, \psi) = \frac{1}{2}\int_{-1}^{1} d\cos\theta \int_0^{4\pi} \frac{d\varphi}{4\pi} \int_0^{4\pi} \frac{d\psi}{4\pi} \, e^{-im\varphi} e^{-im'\psi} d^{(j)}_{mm'}(\theta)$

$$d^{(j)}_{mm'}(\varphi, \theta, \psi) = e^{-im\varphi} e^{-im'\psi} d^{(j)}_{mm'}(\theta)$$

$$\Rightarrow \boxed{\int d\omega \, d^{(j)}_{mm'}(\varphi, \theta, \psi) = \delta_{m,0}\, \delta_{m',0} \frac{1}{2}\int_{-1}^{1} d\cos\theta \, d^{(j)}_{00}(\theta)}$$

but $m = 0 = m' \Rightarrow j \in \mathbb{Z}$ and (for $j \in \mathbb{Z}$)

$$d^{(j)}_{m,0}(\varphi, \theta, \psi) = \sqrt{\frac{4\pi}{2j+1}} \, Y_{jm}(\theta, \varphi)^*$$

since $\frac{1}{2}\int_{-1}^{1} d\cos\theta \int_0^{4\pi} \frac{d\varphi}{4\pi} \sqrt{\frac{4\pi}{2j+1}} \, Y_{jm}(\theta, \varphi)^* = \delta_{j,0}\, \delta_{m,0}$

$$\Rightarrow \boxed{\int d\omega \, d^{(j)}_{mm'}(\varphi, \theta, \psi) = \delta_{j,0}\, \delta_{m,0}\, \delta_{m',0}}$$

$\Rightarrow \int d\omega \, R[\omega] = P_0 \rightarrow$ projection op. onto the singlet state $|0,0\rangle$

These relations can be use to ~~prove~~ show that

$$\int d\omega \; d^{(j_1)}_{m_1' m_1}(\omega) \; d^{(j_2)}_{m_2' m_2}(\omega) = \frac{1}{2j_1+1} \delta_{j_1 j_2} \; \delta_{m_1 m_2} \; \delta_{m_1' m_2'}$$

Let us define the $\underline{\text{character}}$ of the representation $j$ by

$$\chi^{(j)}(\omega) \equiv \operatorname{tr} d^{(j)}(\omega) \equiv \sum_{m=-j}^{j} d^{(j)}_{mm}(\omega)$$

It satisfies

- $\int d\omega \; \chi^{(j)}(\omega) = \delta_{j0}$

- $\int d\omega \; \chi^{(j_1)}(\omega)^* \; \chi^{(j_2)}(\omega) = \delta_{j_1 j_2}$ (orthogonality)

---

Consider now two states $|\lambda j m\rangle$ , $|\lambda' j' m'\rangle$

($\lambda, \lambda'$ are other quantum numbers)

$\Rightarrow \langle \lambda' j' m' | \lambda j m \rangle = 0$ unless $j = j'$ and $m = m'$
(orthonormality)

Let's show that it is independent of $\underline{m}$.

Since $\int d\omega \; R[\omega]^{-1} R[\omega] = I$

$$\Rightarrow \langle \lambda' j' m' | \lambda j m \rangle = \langle \lambda' j' m' | I | \lambda j m \rangle$$
$$= \int d\omega \; \langle \lambda' j' m' | R^{-1}[\omega] R[\omega] | \lambda j m \rangle$$

But

$$R[\omega] |\lambda j m\rangle = \sum_{\bar{m}=-j}^{j} |\lambda j \bar{m}\rangle \; d^{(j)}_{\bar{m} m}(\omega)$$

$$\langle \lambda' j' m' | \lambda j m \rangle = \int d\omega \sum_{\bar{m}, \bar{m}'} d^{(j')}_{\bar{m}' m'}(\omega)^* \; d^{(j)}_{\bar{m} m}(\omega) \; \langle \lambda' j' \bar{m}' | \lambda j \bar{m} \rangle$$

(orthogonality) $$= \delta_{jj'} \; \delta_{mm'} \sum_{\bar{m}} \langle \lambda' j' \bar{m} | \lambda j \bar{m} \rangle \frac{1}{2j'+1}$$

$\Rightarrow \langle \lambda' j' m | \lambda j m \rangle$ is independent of $m$.

Since the state

$$|\tilde{\lambda} j m\rangle = \sum_{q m_1} T_q^{(k)} |\lambda j_1 m_1\rangle \langle k j_1 q m_1 | k j_1 j m\rangle$$

is an eigenstate of $\vec{J}^2$ with ev $j(j+1)\hbar^2$ and of $J_z$ with ev $\hbar m \Rightarrow$

$$\Rightarrow \langle \lambda' j' m' | \tilde{\lambda} j m\rangle = \sum_{q m_1} \langle \lambda' j' m' | T_q^{(k)} | \tilde{\lambda} j_1 m_1\rangle \langle k j_1 q m_1 | k j_1 j m\rangle$$

$$= \delta_{jj'} \delta_{mm'} \sum_{\bar{m}} \frac{\langle \lambda' j' \bar{m} | \tilde{\lambda} j' \bar{m}\rangle}{2j'+1}$$

$$\Rightarrow \sum_{q m_1} \langle \lambda' j' m' | T_q^{(k)} | \lambda j_1 m_1\rangle \langle k j_1 q m_1 | k j_1 j m\rangle =$$

$$= \frac{\delta_{jj'} \delta_{mm'}}{2j'+1} \sum_{\bar{m}} \langle \lambda' j' \bar{m} | \lambda j' \bar{m}\rangle$$

$\Rightarrow$ Using the orthogonality of the CGC's, we find

$$\boxed{\langle \lambda' j' m' | T_q^{(k)} | \lambda j m\rangle = \left( \sum_{\bar{m}} \frac{\langle \lambda' j' \bar{m} | \tilde{\lambda} j' \bar{m}\rangle}{2j'+1} \right) \langle k j q m | k j j' m'\rangle}$$

(Wigner - Eckart Theorem)

(usually written as

$$\langle \lambda' j' m' | T_q^{(k)} | \lambda j m\rangle \equiv \frac{\langle \lambda' j' \| T^{(k)} \| \lambda j\rangle}{\sqrt{2j'+1}} \langle k j q m | k j j' m'\rangle$$

"reduced" matrix element

$$\hookrightarrow \langle \lambda' j' \| T^{(k)} \| \lambda j\rangle \equiv \sum_{\bar{m}} \frac{\langle \lambda' j' \bar{m} | \tilde{\lambda} j \bar{m}\rangle}{\sqrt{2j'+1}}$$

## Applications of the WE Theorem

① Scalar operator $S^{(0)}$ $(k=q=0)$

$$\langle \lambda' j' m' | \cancel{\lambda j} S^{(0)} | \lambda j m \rangle = \delta_{jj'} \delta_{mm'} \frac{\langle \lambda' j \| S_0^{(0)} \| \lambda j \rangle}{\sqrt{2j+1}}$$

since $\langle 0 j 0 m | 0 j j' m \rangle = \delta_{jj'} \delta_{mm'}$

② Matrix elements of $J_q$

$$\langle \lambda' j' m' | J_q | \lambda j m \rangle = \frac{\langle \lambda' j' \| J \| \lambda j \rangle}{\sqrt{2j'+1}} \langle 1 j q m | 1 j j' m' \rangle$$

To evaluate $\langle \lambda' j' \| J \| \lambda j \rangle$ we can choose any $q$

Take $q=0 \Rightarrow J_0 = J_z$

$$\Rightarrow \langle \lambda' j' m' | J_z | \lambda j m \rangle = \hbar m \, \delta_{jj'} \delta_{mm'} \delta_{\lambda\lambda'}$$

$$= \frac{\langle \lambda' j' \| J \| \lambda j \rangle}{\sqrt{2j'+1}} \langle 1 j 0 m | 1 j j' m \rangle$$

$$\langle 1 j 0 m | 1 j j' m \rangle = \frac{m}{\sqrt{j(j+1)}}$$

$$\Rightarrow \frac{\langle \lambda' j' \| J \| \lambda j \rangle}{\sqrt{2j'+1}} = \delta_{jj'} \delta_{\lambda\lambda'} \sqrt{j(j+1)}$$

Finally, even though the product $T^{(k)}_q \; W^{(k')}_{q'}$ is not irreducible, the linear combination $Z^{(j)}_m$

$$Z^{(j)}_m = \sum_{q q'} T^{(k)}_q \; W^{(k')}_{q'} \langle k k' q q' | k k' j m \rangle$$

is irreducible. $\left( \text{i.e. } R \, Z^{(j)}_m R^{-1} = \sum_{m'} Z^{(j)}_{m'} \; d^{(j)}_{m m'} \right)$

and $T^{(k)}_q \; W^{(k')}_{q'} = \sum_{jm} Z^{(j)}_m \langle k k' q q' | k k' j m \rangle$

Let $\vec{U}$, $\vec{V}$ be two vector operators $\Rightarrow$ $U_i V_j$ does not transform simply (i.e. it is reducible). But

$\cancel{V_i U_j = \frac{1}{3} \delta_{ij} (\vec{U}\cdot\vec{V}) + \frac{1}{2}(U_i V_j - V_j U_i)}$

$\cancel{+ \frac{1}{2}(U_i V_j + V_j U_i)}$

$$V_i U_j = \frac{1}{3}(\vec{V} \cdot \vec{U}) + \frac{1}{2}(V_i U_j - V_j U_i) +$$
$$+ \left( \frac{1}{2}(V_i U_j + V_j U_i) - \frac{1}{3} \vec{V}\cdot\vec{U} \, \delta_{ij} \right)$$

$\frac{1}{3} \vec{V}\cdot\vec{U}$ : scalar

$\frac{1}{2}(V_i U_j - V_j U_i) \equiv \varepsilon_{ijk} (\vec{V} \times \vec{U})_k$ is a (pseudo) vector

$\frac{1}{2}(V_i U_j + V_j U_i) - \frac{1}{3} \vec{V}\cdot\vec{U} \, \delta_{ij}$ is a tensor of rank 2.

$\Rightarrow \; Z^{(0)}_0 = -\frac{1}{3} \vec{V}\cdot\vec{U}$ $\qquad Z^{(2)}_{\pm 2} = V_{\pm 1} U_{\pm 1}$

$Z^{(1)}_m = \frac{1}{i\sqrt{2}} (\vec{V} \times \vec{U})_m$ $\qquad Z^{(2)}_{\pm 1} = \frac{1}{\sqrt{2}} (V_{\pm 1} U_0 + V_0 U_{\pm 1})$

$$Z^{(2)}_0 = \frac{1}{\sqrt{6}} (V_1 U_{-1} + 2 V_0 U_0 + V_{-1} U_1)$$