The Dirac Equation

The Klein-Gordon field ϕ describes the dynamics of scalar particles, i.e. particles with spin zero.

~~The~~ Maxwell's theory (when quantized) describes a theory of photons. The field we used is a vector field $A^\mu = (A_0, \vec{A}) \equiv (\Phi, \vec{A})$ which transforms like a 4-vector under Lorentz transformations. We saw that photons carry $S=1$. Can we describe particles with $S = 1/2$?, i.e. electrons, protons, neutrons, etc.? The answer is yes! We need the Dirac field.

Originally the Dirac Equation, and the Dirac field, where developed (by Dirac!) as an alternative to the Klein-Gordon field which at that time (~1930) was believed to be unphysical (now we know it is physical). Dirac wanted to avoid an equation which was second order in time ~~der~~ derivatives (such as the KG equation) in the hope that the ~~result~~ result would be compatible with the requirement of a probabilistic interpretation (i.e. the density > 0) and, in addition of being relativistically covariant, it would not have negative energy solutions. We will see that it is not possible to do all of that but that the answers make sense as a field theory, instead of ~~not~~ ~~a~~ a single particle theory. Since space and time are to be treated on equal footing, if the eqn. is 1^{st} order in $\frac{\partial}{\partial t} \Rightarrow$ it must also be 1^{st} order in $\vec{\nabla}$. In order to avoid violating rotational invariance we must use a multi-component field ψ and write an equation of the form

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar c}{i}\left(\alpha_1 \frac{\partial \psi}{\partial x^1} + \alpha_2 \frac{\partial \psi}{\partial x^2} + \alpha_3 \frac{\partial \psi}{\partial x^3}\right)$$

$$+ \beta m c^2 \psi \equiv H \psi$$

where $\alpha_1, \alpha_2, \alpha_3$ and β are $\overbrace{\text{matrices}}^{N\times N \text{ Hermitean}}$

and ψ is an N-component object (a "spinor")

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar c}{i} \vec{\alpha} \cdot \vec{\nabla} \psi + \beta m c^2 \psi \qquad \left(\begin{array}{c}\text{Dirac}\\ \text{Equation}\end{array}\right)$$

We want this equation to have solutions with

$E^2 = \vec{p}^2 c^2 + m^2 c^4 \Rightarrow$ each component ψ_σ

must satisfy a Klein Gordon Equation

$$\Rightarrow -\hbar^2 \frac{\partial^2 \psi_\sigma}{\partial t^2} = \left(-\hbar^2 c^2 \nabla^2 + m^2 c^4\right) \psi_\sigma$$

$\Rightarrow$ if we iterate the Dirac Eqn we satisfy

this condition if

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \qquad \text{as matrices}$$

$$\{\alpha_i, \beta\} = 0$$

and $\alpha_i^2 = \beta^2 = I$

$$\Rightarrow \quad \operatorname{tr} \alpha_i = \operatorname{tr} \beta = 0$$

Since $\alpha_i^{\dagger} = \alpha_i$ and $\beta^{\dagger} = \beta$, and $\alpha_i^2 = \beta^2 = I$

$\Rightarrow$ their e.v.'s are ± 1. Since $\mathrm{tr}\,\alpha_i = \mathrm{tr}\,\beta = 0$

$\Rightarrow$ they ~~h~~ have the same # of $+1$ ev's

as -1 e.v.'s.

The simplest solution has $N=4$ components (4-spinors)

A particular representation (Dirac basis) is

$$\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

block matrices

Let us find the $\underbrace{\text{density}}_{}$ conserved density and current for this case.

$$i\hbar\, \psi^{\dagger} \frac{\partial \psi}{\partial t} = \frac{\hbar c}{i} \psi^{\dagger} \vec{\alpha}\cdot\vec{\nabla}\psi + mc^2 \psi^{\dagger}\beta\psi$$

$$-i\hbar\, \frac{\partial \psi^{\dagger}}{\partial t}\psi = -\frac{\hbar c}{i} \vec{\nabla}\psi^{*}\cdot\vec{\alpha}\,\psi + mc^2 \psi^{\dagger}\beta\psi$$

since $\vec{\alpha}^{\dagger} = \alpha$ and $\beta^{\dagger} = \beta$

$$\Rightarrow i\hbar \frac{\partial}{\partial t}(\psi^{\dagger}\psi) = \frac{\hbar c}{i} \vec{\nabla}\cdot(\psi^{\dagger}\vec{\alpha}\psi)$$

$\Rightarrow$

if $\rho = \psi^{+}\psi \equiv \sum_{\sigma=1}^{N} \psi_\sigma^{*} \psi_\sigma$

and $\vec{j} = c\ \psi^{+}\ \vec{\alpha}\ \psi \equiv c \sum_{\sigma,\nu=1}^{N} \psi_\sigma^{*} (\vec{\alpha})_{\sigma\nu} \psi_\nu$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \quad ✓$$

$$\Rightarrow \int_V d^3x \left[\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} \right] = 0$$

$$\Rightarrow \frac{\partial}{\partial t} \int d^3x\ \psi^{+}\psi = 0 \quad \text{if} \quad \vec{j}\Big|_{\partial V} = 0$$

If $\psi^{+}\psi > 0 \Rightarrow$ we have a probabilistic interpretation. Superficially this looks promising but we will see that it will fail too.

According ~~the~~ to the requirements of relativistic invariance the density and the current must form a four-vector. Moreover the ~~form of~~ the Dirac Equation must ~~be~~ have ~~of~~ the same form (with the same matrices) in all inertial frames $\Rightarrow$ relativistic covariance solutions: Let $\psi = \psi(t) \Rightarrow$ $\vec{p}=0$ solutions

$$i\hbar \frac{\partial \psi}{\partial t} = \beta m c^2 \psi$$

$$\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$$

$\Rightarrow$ we get <u>four</u> linearly independent solutions

$$\psi^1 = e^{-i\frac{mc^2 t}{\hbar}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \psi^2 = e^{-i\frac{mc^2 t}{\hbar}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Big\}$ <u>2</u> positive energy solutions

$$\psi^3 = e^{+i\frac{mc^2 t}{\hbar}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \psi^4 = e^{+i\frac{mc^2 t}{\hbar}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\Big\}$ <u>2</u> negative energy solutions.

Electromagnetic coupling: $i\hbar \frac{\partial \psi}{\partial t} \to i\hbar \frac{\partial \psi}{\partial t} - e\Phi\psi$

$$c\,\vec{\alpha}\cdot\hat{\vec{p}}\,\psi \to c\,\vec{\alpha}\cdot(\vec{p} - \frac{e}{c}\vec{A})\psi$$

$$\Rightarrow \quad i\hbar \frac{\partial \psi}{\partial t} = \left[c\,\vec{\alpha}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right) + \beta mc^2 + e\Phi\right]\psi$$

$\vec{p} = \frac{\hbar}{i}\vec{\nabla}$ is the equation we want.

Notice that if we write $\Psi = \begin{bmatrix} \bar{\varphi} \\ \bar{\chi} \end{bmatrix}$

where $\bar{\varphi}$ and $\bar{\chi}$ have 2 components

$\Rightarrow$ If we denote by $\vec{\pi} = \vec{p} - \frac{e}{c}\vec{A}$

$$\Rightarrow \quad i\hbar \frac{\partial}{\partial t}\begin{bmatrix} \bar{\varphi} \\ \bar{\chi} \end{bmatrix} = c\,\vec{\sigma}\cdot\vec{\pi}\begin{bmatrix} \bar{\chi} \\ \bar{\varphi} \end{bmatrix} + e\Phi\begin{bmatrix} \bar{\varphi} \\ \bar{\chi} \end{bmatrix} + mc^2\begin{bmatrix} \bar{\varphi} \\ -\bar{\chi} \end{bmatrix}$$

For solution with $E \simeq mc^2 + \cdots$ (i.e. non-relativistic)

$$\begin{bmatrix} \bar{\varphi} \\ \bar{\chi} \end{bmatrix} = e^{-i\frac{mc^2 t}{\hbar}}\begin{bmatrix} \varphi \\ \chi \end{bmatrix}$$

$\Rightarrow$ ~~if we keep the~~

$$i\hbar \frac{\partial}{\partial t}\begin{bmatrix} \varphi \\ \chi \end{bmatrix} = c\,\vec{\sigma}\cdot\vec{\pi}\begin{bmatrix} \chi \\ \varphi \end{bmatrix} + e\Phi\begin{bmatrix} \varphi \\ \chi \end{bmatrix} - 2mc^2\begin{bmatrix} 0 \\ \chi \end{bmatrix}$$

If the energies are small compared to mc^2 we may approximate $\chi \approx \frac{\vec{\sigma}\cdot\vec{\pi}}{2mc}\varphi$

$\Rightarrow \chi$ is a "small" component

and $$i\hbar\frac{\partial\varphi}{\partial t} = \left[\frac{(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi})}{2m} + e\Phi\right]\varphi$$

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot\vec{a}\times\vec{b}$$

$$\Rightarrow (\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi}) = \vec{\pi}^2 + i\vec{\sigma}\cdot\vec{\pi}\times\vec{\pi}$$
$$= \vec{\pi}^2 - \frac{e\hbar}{c}\vec{\sigma}\cdot\vec{B}$$

$$\Rightarrow i\hbar\frac{\partial\varphi}{\partial t} = \left[\frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} + e\Phi\right]\varphi$$

which we recognize as the Pauli Equation

$\Rightarrow$ the components of φ are interpreted as the <u>spin</u> degree of freedom. Notice that the gyromagnetic factor is $g = 2$. and

$$\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$$

The non-relativistic limit of the Dirac Equation

We saw before that if we look at the solutions of the Dirac equation with positive energy (i.e. with eigenvalue close to mc^2) the two lower components form a bi-spinor which is much smaller in magnitude than those of the two upper components (for the negative energy solutions, with eigenvalue close to $-mc^2$, the role of the upper and lower bispinors is revered) This suggests that if one is interested in solutions with energies close to $+mc^2$, it should be possible to systematically eliminate the two lower components and write an effective equation for the upper bispinor. This equation should be of the Pauli form. This is indeed true and we will show how this happens.

The Dirac equation (coupled to an electromagnetic field) has the form

$$i\hbar \frac{\partial \psi}{\partial t} = c\, \vec{\alpha} \cdot \left(\vec{p} - \frac{e}{c}\vec{A}\right) + e\,\Phi + \beta mc^2$$

where Φ is the scalar potential and $\vec{A}$ is the vector potential, and $\vec{p} = \frac{\hbar}{i}\vec{\nabla}$. Here $\vec{\alpha}$ and β are the four Dirac matrices. In the Dirac representation β is diagonal and $\vec{\alpha}$ are off diagonal matrices. We will call the matrices

α^i, $\gamma^i \equiv \beta\alpha^i$, γ_5 <u>odd</u> operators and the matrices $1, \beta, \Sigma$ <u>even</u> operators.

We will use a method known as the Foldy-Wouthuysen transformation. Let U_F be a unitary transformation which we will use to decouple the "large" (upper) from the small (lower) components of a Dirac spinor.

$$\psi' = U_F\, \psi \equiv e^{iS}\psi \quad \text{with } S^{\dagger} = S$$

Ⓐ The free case

$$H = c\,\vec{\alpha}\cdot\vec{p} + \beta mc^2$$

$$\Rightarrow i\hbar \frac{\partial \psi}{\partial t} = H\,\psi \Rightarrow i\hbar \frac{\partial \psi'}{\partial t} = H'\psi'$$

with $H' = e^{iS} H e^{-iS}$

We want ~~S~~ S to be such that H' does not mix upper and lower components, i.e. ~~H~~ H' should not have odd operators. Let us show that

$S = -i\theta\beta\vec{\alpha}\cdot\hat{p}$ works, where $\hat{p} = \dfrac{\vec{p}}{|\vec{p}|}$

$$\Rightarrow e^{iS} = e^{\theta\beta\vec{\alpha}\cdot\hat{p}} = \cos\theta + \beta\vec{\alpha}\cdot\hat{p}\sin\theta$$
$$(\{\beta, \vec{\alpha}\} = 0)$$

$$H' = e^{iS} H e^{-iS} =$$

$$= (\cos\theta + \beta\vec{\alpha}\cdot\hat{p}\sin\theta)(c\,\vec{\alpha}\cdot\vec{p} + mc^2\beta)(\cos\theta - \beta\vec{\alpha}\cdot\hat{p}\sin\theta)$$

$$= (c\,\vec{\alpha}\cdot\vec{p} + \beta mc^2)(\cos\theta - \beta\vec{\alpha}\cdot\hat{p}\sin\theta)^2$$

$$= (c\,\vec{\alpha}\cdot\vec{p} + \beta mc^2)\, e^{-2\beta\vec{\alpha}\cdot\hat{p}\theta}$$

$$= c\,\vec{\alpha}\cdot\vec{p}\left(\cos 2\theta - \frac{mc}{|\vec{p}|}\sin 2\theta\right) + \beta\left(mc^2\cos 2\theta + |\vec{p}|c\sin 2\theta\right)$$

$$\Rightarrow \text{choose} \quad \tan 2\theta = \frac{|\vec{p}|}{mc}$$

$$\Rightarrow H' = \beta\sqrt{m^2c^4 + \vec{p}^2c^2}$$

which clearly has the desired decoupling.

<u>Note 1</u>. This Hamiltonian is non-local

2. It has a simple non-relativistic expansion

$$H' \approx \beta\left[mc^2 + \frac{\vec{p}^{\,2}}{2m} - \frac{\vec{p}^{\,4}}{8m^3c^2} + \cdots\right]$$

Since $\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ $\Rightarrow$ the upper bispinor φ

obeys a Pauli-type equation , $\left(\varphi = \begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}\right)$

$$i\hbar \frac{\partial \varphi}{\partial t} = \left(mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \cdots\right)\varphi$$

(B) <u>The general case</u>

$$H = c\,\vec{\alpha}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right) + \beta mc^2 + e\Phi$$

$$\equiv \beta mc^2 + \underbrace{\mathcal{O}}_{\text{odd}} + \underbrace{\mathcal{E}}_{\text{even}}$$

$$\mathcal{O} = c\,\vec{\alpha}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right)$$

$$\mathcal{E} = e\Phi$$

$$\{\beta, \mathcal{O}\} = 0 \qquad\qquad [\beta, \mathcal{E}] = 0$$

If H is time dependent $\Rightarrow$ S is time-dependent

We will construct S in an expansion in powers of

$\frac{1}{mc^2}$. This means that $|\vec{p}| \ll mc$ (i.e. its matrix elements)

Recall that $\vec{B} = \vec{\nabla} \wedge \vec{A}$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi$$

We will expand to order $\frac{p^4}{m^3c^2}$ and $\frac{\vec{p} \times \vec{E}}{m^2c}, \frac{\vec{p} \times \vec{B}}{m^2 c}$.

$$\Rightarrow \quad H\psi = i\hbar\frac{\partial}{\partial t}\left(e^{-iS}\psi'\right)$$

$$= e^{-iS} i\hbar \frac{\partial \psi'}{\partial t} + \left(i\hbar \frac{\partial}{\partial t} e^{-iS}\right)\psi'$$

$$\Rightarrow \quad i\hbar\frac{\partial \psi'}{\partial t} = \left[e^{iS}\left(H - i\hbar\frac{\partial}{\partial t}\right)e^{-iS}\right]\psi' = H'\psi'$$

$$e^{iS} H e^{-iS} = H + i[S,H] + \frac{i^2}{2!}[S,[S,H]] + \cdots + \frac{i^n}{n!}[S,[S,\cdots[S,H]\cdots]]$$

$$\Rightarrow \quad H' = H + i[S,H] - \frac{1}{2}[S,[S,H]]$$

$H = \beta mc^2 + \cdots$

$$-\frac{i}{6}[S,[S,[S,H]]] + \frac{1}{24}[S,[S,[S,[S,\beta mc^2]]]]$$

$$-\dot{S} - \frac{i}{2}[S,\dot{S}] + \frac{1}{6}[S,[S,\dot{S}]]$$

$$S = O\left(\frac{1}{m}\right)$$

To first order

$$H' = \beta mc^2 + \mathcal{E} + \mathcal{O} + i[S,\beta]mc^2$$

To cancel $\mathcal{O}$, I choose

$$S = \frac{-i\beta\mathcal{O}}{2mc^2}$$

$$i[S,H] = -\mathcal{O} + \frac{\beta}{2mc^2}[\mathcal{O},\mathcal{E}] = \frac{1}{mc^2}\beta\mathcal{O}^2$$

$$\frac{i^2}{2}[S,[S,H]] = -\frac{\beta\mathcal{O}^2}{2mc^2} - \frac{1}{8m^2c^4}[\mathcal{O},[\mathcal{O},\mathcal{E}]] - \frac{1}{2m^2c^4}\mathcal{O}^3$$

$$\frac{i^3}{3!}[S,[S,[S,H]]] = \frac{\mathcal{O}^3}{6m^2c^4} - \frac{1}{6m^3c^6}\beta\mathcal{O}^4$$

$$\frac{i^4}{4!}[S,[S,[S,[S,H]]]] = \frac{\beta\mathcal{O}^4}{24m^3c^6}$$

$$-\dot{S} = \frac{i\beta\dot{\mathcal{O}}}{2mc^2}$$

$$-i[S,\dot{S}] = \frac{-i}{8m^2c^4}[\mathcal{O},\dot{\mathcal{O}}]$$

$\Rightarrow$

$$H' = \beta\left(mc^2 + \frac{\mathcal{O}^2}{2mc^2} - \frac{\mathcal{O}^4}{8m^3c^6}\right) + \mathcal{E} - \frac{1}{8m^2c^4}[\mathcal{O},[\mathcal{O},\mathcal{E}]]$$

$$- \frac{i\hbar}{8m^2c^4}[\mathcal{O},\dot{\mathcal{O}}] + \frac{\beta}{2mc^2}[\mathcal{O},\mathcal{E}] - \frac{\mathcal{O}^3}{3m^2c^4} + \frac{i\beta\hbar\dot{\mathcal{O}}}{2mc^2}$$

$$\equiv \beta mc^2 + \mathcal{E}' + \mathcal{O}'$$

Now $\mathcal{O}' = O\left(\frac{1}{mc^2}\right) \Rightarrow$ we can transform H' by S' to cancel $\mathcal{O}'$

$$S' = \frac{-i\beta}{2mc^2}\mathcal{O}' = \frac{-i\beta}{2mc^2}\left(\frac{\beta}{2mc^2}[\mathcal{O},\mathcal{E}] - \frac{\mathcal{O}^3}{3m^2c^4} + \frac{i\beta\hbar\dot{\mathcal{O}}}{2mc^2}\right)$$

$$\Rightarrow \quad H'' = e^{iS'}\left(H' - i\hbar\frac{\partial}{\partial t}\right)e^{iS'}$$

$$= \beta mc^2 + \mathcal{E}' + \frac{\beta}{2mc^2}[\mathcal{O}', \mathcal{E}'] + \frac{i\beta\hbar\dot{\mathcal{O}}'}{2mc^2}$$

$$= \beta mc^2 + \mathcal{E}' + \mathcal{O}''$$

where $\mathcal{O}'' \sim \frac{1}{(mc^2)^2}$ which can be cancelled by

$$S'' = \frac{-i\beta\mathcal{O}''}{2mc^2} \Rightarrow H'''$$

$$H''' = e^{iS''}\left(H'' - i\hbar\frac{\partial}{\partial t}\right)e^{-iS''} = \beta mc^2 + \mathcal{E}'$$

$$= \beta\left(mc^2 + \frac{\mathcal{O}^2}{2mc^2} - \frac{\mathcal{O}^4}{8m^3c^6}\right) + \mathcal{E}$$

$$- \frac{1}{8m^2c^4}[\mathcal{O},[\mathcal{O},\mathcal{E}]] - \frac{i\hbar}{8m^2c^4}[\mathcal{O},\dot{\mathcal{O}}]$$

$$\Rightarrow \frac{\mathcal{O}^2}{2mc^2} = \frac{\cancel{c^2}}{2m\cancel{c^2}} \left(\vec{\alpha}\cdot\left(\vec{p}-\frac{e}{c}\vec{A}\right)\right)^2$$

$$= \frac{1}{2m}\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2 - \frac{e\hbar}{2mc}\vec{\Sigma}\cdot\vec{B}$$

$$\frac{1}{8m^2c^4}\left([\mathcal{O},\mathcal{E}] + i\hbar\dot{\mathcal{O}}\right) = \frac{e\hbar}{8m^2c^3}\left(-i\,\vec{\alpha}\cdot\vec{\nabla}\Phi - \frac{i}{c}\,\vec{\alpha}\cdot\frac{\partial\vec{A}}{\partial t}\right)$$

$$= \frac{ie\hbar}{8m^2c^3}\,\vec{\alpha}\cdot\vec{E}$$

$$\left[\mathcal{O}, \frac{ie\hbar}{8m^2c^3}\vec{\alpha}\cdot\vec{E}\right] = \frac{ie\hbar}{8m^2c^2}\left[\vec{\alpha}\cdot\vec{p}, \vec{\alpha}\cdot\vec{E}\right]$$

$$= \frac{e\hbar^2}{8m^2c^2}\vec{\nabla}\cdot\vec{E} + i\frac{e\hbar^2}{8m^2c^2}\vec{\Sigma}\cdot\vec{\nabla}\times\vec{E}$$

$$+ \frac{e\hbar^2}{4m^2c^2}\vec{\Sigma}\cdot\vec{E}\times\vec{p}$$

$$H''' = \beta\left[mc^2 + \frac{1}{2m}\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2 - \frac{p^4}{8m^3c^2}\right]$$

$$+ e\Phi - \frac{e\hbar}{2mc}\vec{\Sigma}\cdot\vec{B}$$

$$- \frac{e\hbar^2}{8m^2c^2}\vec{\nabla}\cdot\vec{E} - \frac{ie\hbar^2}{8m^2c^2}\vec{\Sigma}\cdot(\vec{\nabla}\times\vec{E}) - \frac{e\hbar}{4m^2c^2}\vec{\Sigma}\cdot\vec{E}\times\vec{p}$$

$$H_{spin-orbit} = \frac{-ie\hbar^2}{8m^2c^2}\left(\vec{\Sigma}\cdot\vec{\nabla}\times\vec{E}\right) - \frac{e\hbar}{4m^2c^2}\left(\vec{\Sigma}\cdot\vec{E}\times\vec{p}\right)$$

For a static spherically symmetric potential Φ

$$\vec{\Sigma}\cdot\vec{E}\times\vec{p} = -\frac{1}{r}\frac{\partial\Phi}{\partial r}\vec{\Sigma}\cdot\vec{r}\times\vec{p}$$

$$= -\frac{1}{r}\frac{\partial\Phi}{\partial r}\vec{\Sigma}\cdot\vec{L}$$

and $\vec{\nabla}\times\vec{E} = 0$ $\qquad$ $\vec{\nabla}\cdot\vec{E} = 4\pi Z|e|\,\delta^3(\vec{r})$

$$\Rightarrow H_{spin-orbit} = \frac{e\hbar}{4m^2c^2}\frac{1}{r}\frac{\partial\Phi}{\partial r}\vec{\Sigma}\cdot\vec{L}$$

$$H_{Darwin} = -\frac{e\hbar^2}{8m^2c^2}\vec{\nabla}\cdot\vec{E} = -\frac{e\hbar^2\pi}{8m^2c^2}Z|e|\,\delta^3(r)$$

$$= +\frac{Z\alpha}{2m^2}\delta^3(r)$$