Relativistic Equations and Lorentz Transformations:

Let us denote by $x^\mu = (ct, \vec{x})$ a

contravariant 4-vector and by $x_\mu = (ct, -\vec{x})$
$(x_0 = x^0 = ct)$
a covariant 4-vector. They are related by the action of the metric tensor of Minkowski space

$$x^\mu = g^{\mu\nu} x_\nu \qquad \text{(repeated indices are summed)}$$

$$g^{\mu\nu} \equiv g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

whereas

$$g^\mu{}_\nu \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \delta^\mu{}_\nu$$

Energy-Momentum 4-vector: $p^\mu = \left(\frac{E}{c}, \vec{p}\right)$

Scalar product: $a^\mu b_\mu = a_0 b_0 - \vec{a} \cdot \vec{b}$

gradients: $\frac{\partial}{\partial x^\mu} \equiv \partial_\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla} \right)$ covariant

$\frac{\partial}{\partial x_\mu} \equiv \partial^\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, -\vec{\nabla} \right)$ contravariant

(note the sign change)

$\Rightarrow \quad \partial_\mu j^\mu = \frac{1}{c} \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j}$

$j^\mu = (\rho, \vec{j})$

$\Rightarrow$ continuity $\Longleftrightarrow$ $\partial_\mu j^\mu = 0$

D'Alambertian: $\Box \equiv \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$

Relativistic Interval: $x^2 \equiv x_\mu x^\mu = c^2 t^2 - \vec{x}^2$

L29 <u>Lorentz Transformations</u>: Linear transformations

$x'^\mu = \Lambda^\mu{}_\nu x^\nu$ which leave x^2 invariant

$\Rightarrow$ i.e. $c^2 t'^2 - \vec{x}'^2 = c^2 t^2 - \vec{x}^2$

They consist of 3 Lorentz boosts + 3 rotations.

Generator of infinitesimal rotations in $D=3$ dimensions is the tensor $J^{ij} = -i \left(x^i \nabla^j - x^j \nabla^i \right)$

$\Rightarrow$ Generator of infinitesimal Lorentz transformations is

$J^{\mu\nu} = i \left(x^\mu \partial^\mu - x^\nu \partial^\nu \right)$

which obey the algebra

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\,(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho})$$

In particular the 4×4 matrices

$$(J^{\mu\nu})_{\alpha\beta} = i\,(\delta^{\mu}{}_{\alpha}\,\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}\,\delta^{\nu}{}_{\alpha})$$

obey this algebra.

$$V^{\alpha} \rightarrow V'^{\alpha} = V^{\alpha} + \delta V^{\alpha}$$

$$\delta V^{\alpha} = -\frac{i}{2}\,\omega_{\mu\nu}(J^{\mu\nu})^{\alpha}{}_{\beta}\,V^{\beta}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are the infinitesimal Euler angles

e.g. if $\omega_{12} = -\omega_{21} = \theta \neq 0$ (all other $=0$)

$$\Rightarrow V \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} V$$

i.e. an infinitesimal rotation.

if $\omega_{01} = -\omega_{10} = \beta$

$$\Rightarrow V \rightarrow \begin{pmatrix} 1 & \beta & 0 & 0 \\ \beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} V$$

is an infinitesimal Lorentz boost

Dirac γ- matrices: γ^μ

$$\gamma^0 = \beta \qquad \text{and} \qquad \gamma^i = \beta\alpha^i$$

$$\Rightarrow \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \text{and} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$\Rightarrow$ we get

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$\leftarrow$ Minkowski metric tensor.

Feynman's slash: $\not{a} = a_\mu \gamma^\mu$

$\Rightarrow$ ~~[illegible]~~ $\qquad (\gamma_j = -\gamma^j)$

$$\left(i\hbar \frac{\partial}{\partial t} - \frac{\hbar c}{i}\beta\gamma_j \frac{\partial}{\partial x^j} - \beta mc^2\right)\psi = 0$$

$$\Rightarrow \left[i\left(\frac{\beta}{c}\frac{\partial}{\partial t} + \gamma_j \frac{\partial}{\partial x^j}\right) - \frac{mc}{\hbar}\right]\psi = 0$$

$$\Rightarrow \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - \frac{mc}{\hbar}\right)\psi = 0$$

$$\text{or } \left(i\not{\partial} - \frac{mc}{\hbar}\right)\psi = 0$$

which <u>looks</u> Lorentz invariant. We need to check ~~w~~ how does ψ transform.

Note: the current and density ~~for~~m a 4-vector

$$j^\mu = (j^0, \vec{j})$$

$$j^0 = \psi^+ \psi \quad , \quad \vec{j} = \psi^+ \vec{\alpha}\, \psi$$

$$\bar{\psi} \equiv \psi^+ \gamma^0 \Rightarrow \boxed{j^\mu = \bar{\psi} \gamma^\mu \psi}$$

and $(i\not{\partial} - m)\, \psi = 0 \Rightarrow \bar{\psi}\,(i\overleftarrow{\not{\partial}} + m) = 0$

Relativistic Covariance $\Rightarrow$ $x' = \Lambda x$ is a L.T.

$$\Rightarrow i\gamma^\mu \partial_\mu \psi - m\psi = 0 \Rightarrow i\gamma^\mu \partial'_\mu \psi'(x') - m\, \psi'(x) = 0$$

with the <u>same</u> Dirac matrices and the <u>same</u> mass

$$\psi'(x') = S(\Lambda)\, \psi(x) \qquad \text{for } x' = \Lambda x$$

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = (\Lambda^{-1})^\nu{}_\mu \frac{\partial}{\partial x^\nu}$$

$$\Rightarrow i\gamma^\mu \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu}\left[S(\Lambda)\, \psi(x)\right] - m\, S(\Lambda)\, \psi(x) = 0$$

$$i\gamma^\mu (\Lambda^{-1})^\nu{}_\mu\, S(\Lambda)\, \partial_\nu \psi - m\, S(\Lambda)\, \psi = 0$$

$$\Rightarrow i\, S(\Lambda)^{-1} \gamma^\mu S(\Lambda)\, (\Lambda^{-1})^\nu{}_\mu\, \partial_\nu \psi - m\psi = 0$$

$$\Rightarrow \quad S(\Lambda)^{-1} \gamma^\mu S(\Lambda)\, (\Lambda^{-1})^\nu{}_\mu = \gamma^\nu$$

with $(S(\Lambda))^{-1} = S(\Lambda^{-1})$

$$\Rightarrow \quad S(\Lambda)^{-1}\gamma^{\mu} S(\Lambda) = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}$$

Let us construct $S(\Lambda)$ for an infinitesimal Lorentz transformation

$$\Lambda^{\mu}{}_{\nu} = \left(I - \frac{i}{2}\omega_{\rho\sigma} J^{\rho\sigma}\right)^{\mu}{}_{\nu}$$

and $\quad S(\Lambda) = I - \frac{i}{2}\omega_{\alpha\beta}\sigma^{\alpha\beta}$

with $\sigma^{\alpha\beta} = -\sigma^{\beta\alpha}$ and $\omega_{\mu\nu} = -\omega_{\nu\mu}$

$$\Rightarrow \left(I + \frac{i}{2}\omega\cdot\sigma\right)\gamma^{\mu}\left(I - \frac{i}{2}\omega\cdot\sigma\right) = \left(I - \frac{i}{2}\omega\cdot J\right)^{\mu}{}_{\nu}\gamma^{\nu}$$

$$\Rightarrow \boxed{[\gamma^{\mu}, \sigma^{\alpha\beta}] = (J^{\alpha\beta})^{\mu}{}_{\nu}\gamma^{\nu}}$$

i.e. $\quad [\gamma^{\mu}, \sigma^{\alpha\beta}] = i\left(g^{\mu\alpha} g^{\beta}{}_{\nu} - g^{\alpha}{}_{\nu} g^{\mu\beta}\right)\gamma^{\nu}$

$$\Rightarrow \quad [\gamma^{\mu}, \sigma^{\alpha\beta}] = i\left(g^{\mu\alpha}\gamma^{\beta} - g^{\mu\beta}\gamma^{\alpha}\right)$$

$\Rightarrow$ the solution is

$$\sigma^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]$$

~~A solution is~~ $\cancel{\sigma_{\alpha\beta} = \frac{i}{4}[\gamma_\alpha, \gamma_\beta]}$

$\Rightarrow$ Finite Transformations

$$S(\Lambda) = e^{-\frac{i}{4}\sigma_{\alpha\beta}\,\omega^{\alpha\beta}}$$

where $\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}}$ ($J^{\mu\nu}$ defined before)

$\Rightarrow \quad x' = \Lambda x \quad \Rightarrow \quad \psi'(x') = S(\Lambda)\,\psi(x)$

[note:

$$\sigma^{jk} = \frac{1}{2}\varepsilon^{jkl}\begin{bmatrix}\sigma_l & 0 \\ 0 & \sigma_l\end{bmatrix} = \frac{i}{4}\left[\gamma^j, \gamma^k\right] \equiv \frac{1}{2}\varepsilon^{ijk}\Sigma^k$$

$$\Sigma^k = \begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix}$$

$$\sigma^{0j} = \frac{i}{4}\left[\gamma^0, \gamma^j\right] = \frac{i}{2}\begin{bmatrix}\sigma^j & 0 \\ 0 & -\sigma^j\end{bmatrix}$$

$$\sigma^{jk} = \sigma_{jk} \quad \text{and} \quad \sigma^{0j} = -\sigma_{0j}$$]

$\Rightarrow S(\Lambda)$ yields the field in the transformed frame in terms of the coordinates of the transformed frame.

Q: What is the unitary transformation $U(\Lambda)$ that just compensates the effect of the coordinate transf.?

$$\psi'(x) = U(\Lambda)\,\psi(x) = S(\Lambda)\,\psi(\Lambda^{-1}x)$$

$$\Rightarrow \quad U(\Lambda) = I - \frac{i}{2} J_{\mu\nu}\, \omega^{\mu\nu} + \cdots$$

$$\left(I - \frac{i}{2} J_{\mu\nu}\, \omega^{\mu\nu}\right) \psi(x) = \left(I - \frac{i}{2} \sigma_{\mu\nu}\, \omega^{\mu\nu}\right) \psi(x^\rho - \omega^\rho{}_\nu\, x^\nu)$$

$$= \left(I - \frac{i}{2} \sigma_{\mu\nu}\, \omega^{\mu\nu}\right) \left(\psi(x) - \omega^\rho{}_\nu\, x^\nu\, \partial_\rho \psi\right)$$

$$\Rightarrow \psi'(x) = \left(I - \frac{i}{2} \sigma_{\mu\nu}\, \omega^{\mu\nu} + x_\mu\, \omega^{\mu\nu}\, \partial_\nu\right) \psi(x)$$

$$\Rightarrow \quad \boxed{J_{\mu\nu} = \sigma_{\mu\nu} + i\,(x_\mu \partial_\nu - x_\nu \partial_\mu)}$$

intrinsic angular momentum

generalized angular momentum

In particular

$$J_{jk} = i\,(x_j \nabla_k - x_k \nabla_j) + \frac{1}{2} \varepsilon_{jkl} \Sigma^l$$

$$\Sigma^l = \begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{pmatrix} \equiv \sigma^l$$

Define a 3-component angular momentum

$$J_i = \frac{1}{2} \varepsilon_{ijk} J_{jk} \Rightarrow$$

$$\vec{\Sigma}$$

$$\vec{J} = \vec{x} \wedge \vec{p} + \frac{\hbar}{2} \vec{\sigma} \quad \equiv \frac{\hbar}{i} \vec{x} \wedge \vec{\nabla} + \frac{\hbar}{2} \vec{\sigma}$$

orbital spin !

$\Rightarrow$ Dirac spinors carry spin 1/2 !

Solutions of the Dirac Equation

We seek plane wave solutions $(\hbar = c = 1)$

$$\psi^{(+)}(x) = e^{-i p\cdot x}\, u(k) \qquad \text{pos. energy}$$

$$\psi^{(-)}(x) = e^{i p\cdot x}\, v(k) \qquad \text{neg. energy}$$

s.t. $p^0 > 0$ $\qquad$ p^μ is a 4-vector. (energy-momentum)

$\Rightarrow$ $\cancel{(\not{p} - m)\psi = 0}$ $(i\not{\partial} - m)\psi = 0$ is the Dirac Equation.

$$\Rightarrow \quad (\not{p} - m)\, u(p) = 0 \qquad + \text{ energy}$$

$$(\not{p} + m)\, v(p) = 0 \qquad - \text{ energy}$$

Rest frame $(m \neq 0)$

$$\Rightarrow \quad (\gamma^0 - I)\, u(m, \vec{0}) = 0 \qquad p^\mu = (m, 0)$$

$$(\gamma^0 + I)\, v(m, \vec{0}) = 0$$

$$\Rightarrow \quad u^{(1)}(m, \vec{0}) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad u^{(2)}(m, \vec{0}) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v^{(1)}(m, \vec{0}) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v^{(2)}(m, \vec{0}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

note: $(\not{p}+m)(\not{p}-m) = p^2 - m^2$

$$\Rightarrow \quad u^{(\alpha)}(p) = \frac{(\not{p}+m)}{[2m(E+m)]^{1/2}} \, u^{(\alpha)}(m,\vec{0})$$

$$= \begin{bmatrix} \left(\frac{E+m}{2m}\right)^{1/2} \varphi^{(\alpha)}(m,\vec{0}) \\ \frac{\vec{\sigma}\cdot\vec{p}}{[2m(E+m)]^{1/2}} \varphi^{(\alpha)}(m,\vec{0}) \end{bmatrix}$$

$$E = \sqrt{p^2+m^2} \equiv \sqrt{p^2c^2+m^2c^4}$$

$$v^{(\alpha)}(p) = \frac{(-\not{p}+m)}{(2m(E+m))^{1/2}} \, v^{(\alpha)}(m,\vec{0})$$

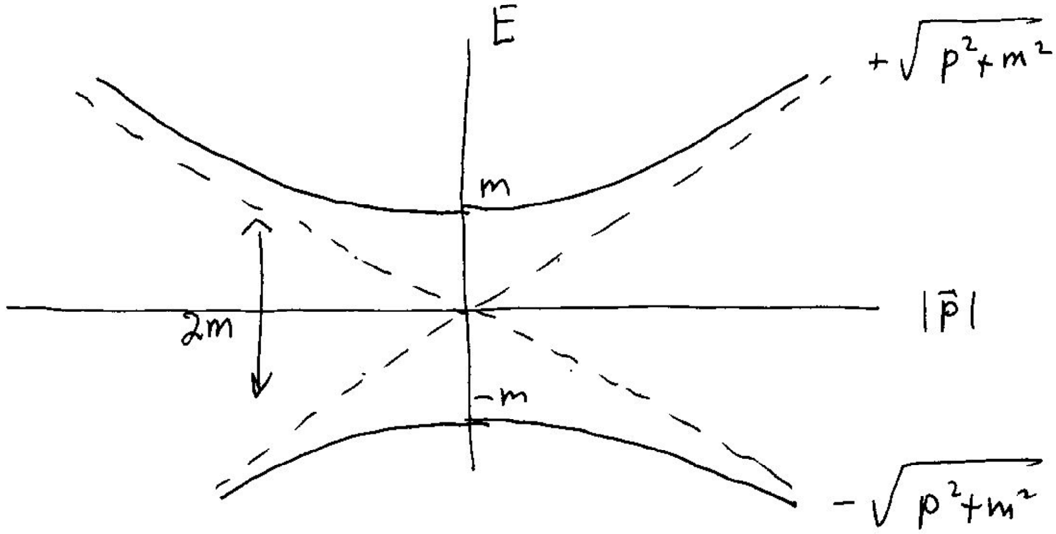
$$= \begin{bmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{[2m(E+m)]^{1/2}} & \chi^{(\alpha)}(m,\vec{0}) \\ \left(\frac{E+m}{2m}\right)^{1/2} & \chi^{(\alpha)}(m,\vec{0}) \end{bmatrix}$$

$$\Rightarrow \quad \psi^{(+)}_{p,\alpha}(x) = e^{-ip\cdot x} \, u^{(\alpha)}(p) \qquad + \text{ energy}$$

$$\psi^{(-)}_{p,\alpha}(x) = e^{ip\cdot x} \, v^{(\alpha)}(p) \qquad - \text{ energy.}$$

and $i\,\partial_t \psi^{(+)}_p(x) = p_0 \psi^{(+)}_p(x) = +E \, \psi^{(+)}_p(x)$

$i\,\partial_t \psi^{(-)}_p(x) = -p_0 \psi^{(-)}_p(x) = -E \, \psi^{(-)}_p(x)$



note on normalization:

$$\bar{u}^{\beta}(p)\; u_{\alpha}(p) = \delta^{\alpha\beta}\; \varepsilon_{\alpha} \qquad \varepsilon_{\alpha} = \begin{cases} 1 & \alpha = 1,2 \\ -1 & \alpha = 3,4 \end{cases}$$

$$\bar{u} \equiv u^{\dagger}\gamma^{0}$$

and

$$\bar{u}^{\alpha}(p)\; u^{\beta}(p) = \delta_{\alpha\beta}$$

$$\bar{v}^{\alpha}(p)\; v^{\beta}(p) = -\,\delta_{\alpha\beta}$$

$$\bar{u}^{\alpha}(p)\; v^{\beta}(p) = 0$$

$$\bar{v}^{\alpha}(p)\; u^{\beta}(p) = 0$$

Plane wave expansions

We want to construct wave packets of these plane waves. Notice that the negative energy solutions are necessary to have a complete basis of eigenvectors.

Let $\psi^{(+)}(x)$ be an arbitrary linear combination of (+) energy solutions

$$\psi^{(+)}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{m}{E(p)} \sum_{\alpha=1,2} \underbrace{b(\alpha,p)}_{\text{amplitudes}} u^{(\alpha)}(p)\, e^{-ip\cdot x}$$

s.t.

$$\int d^3x\, \psi^{(+)}(x)^{\dagger}\, \psi^{+}(x) = \int d^3x\, j^{(+)0}(x,t) = 1$$

note: $\int \frac{d^3p}{E(p)}$ is Lorentz invariant

Suppose the wave packet at $t=0$ is

$$\psi(0,\vec{x}) = \frac{1}{(\pi d^2)^{3/4}}\, e^{-\frac{\vec{x}^2}{2d^2}}\, w$$

$w = \begin{pmatrix} \varphi \\ 0 \end{pmatrix}$ i.e. a (+) energy Gaussian wave packet

$$\Rightarrow \Psi(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{m}{E(p)} \sum_\alpha \Big(b(p,\alpha)\, u^{(\alpha)}(p)\, e^{-ip\cdot x} + d^*(p,\alpha)\, v^{(\alpha)}(p)\, e^{ip\cdot x} \Big)$$

Since

$$\int d^3x \; e^{-\frac{\vec{x}^2}{2d} - i\vec{p}\cdot\vec{x}} = \frac{1}{(2\pi d^2)^{3/2}} e^{-\frac{\vec{p}^2 d^2}{2}}$$

$$\Rightarrow (4\pi d^2)^{3/4} e^{-\frac{\vec{p}^2 d^2}{2}} w = \frac{m}{E} \sum_\alpha \Big(b(p,\alpha)\, u^\alpha(p) + d^*(\tilde{p},\alpha)\, v^\alpha(\tilde{p}) \Big)$$

$$\tilde{p} = (p^0, -\vec{p})$$

$$\Rightarrow \quad b(p,\alpha) = (4\pi d^2)^{3/4} e^{-\frac{\vec{p}^2 d^2}{2}} u^{(\alpha)}(p)^+ w$$

$$d^*(p,\alpha) = (4\pi d^2)^{3/4} e^{-\frac{\vec{p}^2 d^2}{2}} v^{(\alpha)}(p)^+ w$$

$\Rightarrow$ there is always a contribution of the negative energy states.

$$\Rightarrow \quad u^{(\alpha)+} w = \left(\frac{E+m}{2m}\right)^{1/2}$$

$$v^{(\alpha)+} w = \varphi^+ \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{2m(E+m)}} \varphi \equiv \frac{p}{(2m(E+m))^{1/2}}$$

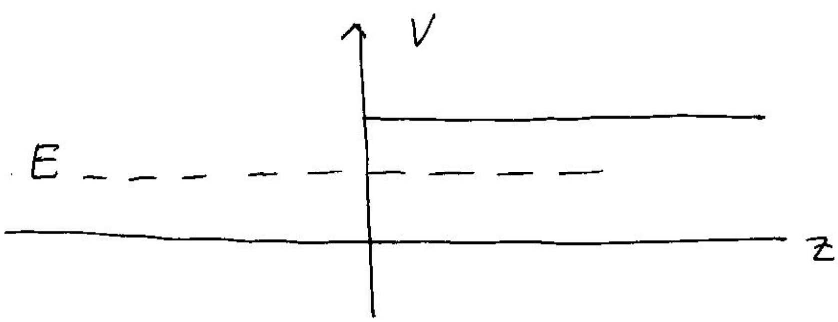
$$\Rightarrow \quad \frac{b}{d^*} \sim \frac{E+m}{|\vec{p}|}$$

$$\Rightarrow \quad \frac{d^*}{b} \sim \frac{|\vec{p}|}{E+m}$$ is important for $|\vec{p}| \sim m \sim \frac{1}{d}$

$\Rightarrow$ if $d \gg \frac{1}{m}$ $\Rightarrow$ 1 particle theory is consistent

for $d \ll \frac{1}{m}$ $\Rightarrow$ 1 particle theory breaks down.

<u>Klein Paradox</u>:



$p^\mu = (p^0, 0, 0, p)$

$$\psi_{inc}(z) \simeq e^{ipz} \begin{pmatrix} 1 \\ 0 \\ p/(E+m) \\ 0 \end{pmatrix}$$ spin up along the z axis

$$\psi_{ref}(z) = a\, e^{-ipz} \begin{pmatrix} 1 \\ 0 \\ -p/(E+m) \\ 0 \end{pmatrix} +$$ $z<0$

$$+\, b\, e^{-ipz} \begin{pmatrix} 0 \\ 1 \\ 0 \\ p/(E+m) \end{pmatrix}$$ ↑ and ↓ with + energy

$z > 0$

$$\psi_{trans}(z) = c\, e^{iqz} \begin{pmatrix} 1 \\ 0 \\ q/(E-V+m) \\ 0 \end{pmatrix} + d\, e^{-iqz} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -q/(E-V+m) \end{pmatrix}$$

$$q = \sqrt{(E-V)^2 - m^2}\ , \qquad E = \sqrt{p^2 + m^2}$$

Continuity $\Rightarrow$ (no spin flip) $\quad b = d = 0$

$$1 + a = c$$

$$1 - a = rc$$

$$r = \frac{q}{p}\left(\frac{E+m}{E-V+m}\right) \qquad \text{reflection coefficient}$$

If $|E-V| < m \Rightarrow q$ is imaginary and the transmitted wave decays (exponentially), but

for $V \geq E+m \Rightarrow$ oscillating transmitted wave

but $r = \dfrac{q}{p}\,\dfrac{E+m}{E-V+m} < 0$

$$\Rightarrow \frac{j_{ref}}{j_{inc}} = \left(\frac{1-r}{1+r}\right)^2 = 1 - \frac{j_{trans}}{j_{inc}} > 1 \ ! \qquad \underline{\text{pair}}\ \ \underline{\sim\sim\text{creation}\sim\sim}\ \underline{\text{creation}}$$

$\Rightarrow$ if we want to localize a particle in $d \lesssim \frac{1}{m} \Rightarrow q_t <$

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The Dirac Field

We still need to make sense of the negative energy solutions. We will proceed in a manner similar to what we did in the scalar field case but with an important difference: we will use anticommutators.

Our first task is to find a Hamiltonian $\hat{H}$ for the field theory of the Dirac equation. $\hat{H}$ must be such that

$$i\,\partial_t \psi = [\psi, H]$$ must be equivalent to the Dirac Equation.

Let $$\hat{H} = \int d^3x \;\psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + m\beta)\psi$$

$$\equiv \int d^3x \;\bar{\psi}\,\underbrace{(-i\vec{\gamma}\cdot\vec{\nabla} + m)}_{\text{one-particle Hamiltonian}}\psi$$

$$\bar{\psi} = \psi^\dagger\gamma^0 = \psi^\dagger\beta$$

It is easy to ~~say~~ check that the equation of motion

$$i\gamma_0 \partial_0 \psi = [\gamma_0 \psi, H] = (-i\,\vec{\gamma}\cdot\vec{\nabla} + m)\,\psi$$

holds independently of the choice of statistics, i.e. whether the operators $\psi_\alpha(x)$ $(\alpha = 1, \ldots, 4)$ obey equal-time commutation or anticommutation relations.

We will solve the equation of motion, i.e. the Dirac equation, by expanding in modes

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{m}{\omega(p)} \left(\tilde{\psi}_+(p)\, e^{-ip\cdot x} + \tilde{\psi}_-(p)\, e^{+ip\cdot x} \right)$$

where $\omega(p) = p_0 = \sqrt{\vec{p}^{\,2} + m^2} = E$, and

$$p\cdot x = p_0 x_0 - \vec{p}\cdot\vec{x} = E\,t - \vec{p}\cdot\vec{x}$$

where $\tilde{\psi}_\pm(p)$ obey

$$(p_0\gamma_0 - \vec{\gamma}\cdot\vec{p} \pm m)\,\tilde{\psi}_\pm(p) = 0$$

$\Rightarrow$ if $\tilde{\psi}_\pm(p) = (\pm \not{p} + m)\,\tilde{\phi}$

$(\not{p} = p_\mu \gamma^\mu)$

$$\Rightarrow \quad (\not{p} \mp m)(\not{p} \pm m)\,\tilde{\phi} = \pm\,(p^2 - m^2)\,\tilde{\phi} = 0$$

$\Rightarrow$ we demand $p^2 = m^2$ and $\tilde{\phi}$ is an arbitrary 4-spinor at $\vec{p}=0$ $\qquad \tilde{\Psi}_{\pm}(p_0, \vec{p}=0) = (\pm p_0 \gamma_0 + m)\,\tilde{\phi}$

I choose $\tilde{\phi}$ to be an eigenstate of γ_0

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$\Rightarrow$ $\quad u^{(1)}(m,\vec{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\quad u^{(2)}(m,\vec{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

have γ_0 eigenvalue $+1$

and $\quad v^{(1)}(m,\vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\quad v^{(2)}(m,\vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

have γ_0 eigenvalue -1

$\varphi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\varphi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\tilde{\Psi}_{+}(p) = \frac{\not{p}+m}{\sqrt{2m(p_0+m)}}\, u^{(\sigma)}(m,\vec{0}) = \begin{pmatrix} \sqrt{\frac{p_0+m}{2m}}\, \varphi^{(\sigma)}(m,\vec{0}) \\ \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{2m(p_0+m)}}\, \varphi^{(\sigma)}(m,\vec{0}) \end{pmatrix}$$

as before

$$\tilde{\Psi}_{-}(p) = \frac{-\not{p}+m}{\sqrt{2m(p_0+m)}}\, v^{(\sigma)}(m,\vec{0})$$

We can now write the mode expansion as

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \sum_{\sigma=1,2} \Big[\hat{a}_{\sigma,+}(\vec{p})\, u^{(\sigma)}(\vec{p})\, e^{-ip\cdot x} + \hat{a}_{\sigma,-}(\vec{p})\, v^{(\sigma)}(\vec{p})\, e^{ip\cdot x} \Big]$$

where the operators $\hat{a}_{\sigma,\pm}(\vec{p})$ obey as-yet unspecified commutation relations.

The (formal) Hamiltonian is

$$H = \int \frac{d^3p}{(2\pi)^3} \left(\frac{m}{p_0}\right) \sum_{\sigma=1,2} p_0 \Big[\hat{a}^{+}_{\sigma,+}(p)\, \hat{a}_{\sigma,+}(\vec{p}) - \hat{a}^{+}_{\sigma,-}(p)\, \hat{a}_{\sigma,-}(\vec{p}) \Big]$$

$\Rightarrow$ the $(+)$ modes have energy $p_0 = \sqrt{p^2+m^2}$ and the $(-)$ modes have energy $p_0 = -\sqrt{p^2+m^2}$

$\Rightarrow$ since the single particle spectrum does not have a lower bound $\Rightarrow$ the spectrum of H is <u>not</u> bounded from below if the operators $\hat{a}_{\sigma,\pm}(p)$ obey canonical <u>commutation</u> relations $\Rightarrow$ bosons with a Dirac spectrum have no ground state!

Dirac solved this problem by demanding that these excitations obey the Pauli principle $\Rightarrow$ $S=1/2$ fields must be quantized with anticommutators and are fermions.

$$\Rightarrow \quad \{ \hat{a}^{\dagger}_{\sigma,s}(\vec{p}) , \hat{a}_{s',\sigma'}(\vec{p}\,') \} = 0 \qquad s,s = \pm$$

$$\{ \hat{a}^{\dagger}_{\sigma,s}(\vec{p}) , \hat{a}_{\sigma',s'}(\vec{p}\,') \} = (2\pi)^3 \frac{p_0}{m} \delta^3(\vec{p}-\vec{p}\,') \delta_{\sigma\sigma'} \delta_{ss'}$$

let $|0\rangle$ be the state annihilated by $\hat{a}_{\sigma,s}(\vec{p})$

$$\hat{a}_{\sigma,s}(p)|0\rangle = 0$$

We will now see that $|0\rangle$ is <u>not</u> the vacuum.

Let $\quad |vac\rangle \equiv \prod_{\sigma,\vec{p}} \hat{a}^{\dagger}_{\sigma,-}(\vec{p})\,|0\rangle$

i.e. we fill up all negative energy states.

and define

$$\hat{b}_{\sigma}(\vec{p}) = \hat{a}_{\sigma,+}(\vec{p})$$

$$\hat{d}_{\sigma}(\vec{p}) = \hat{a}^{\dagger}_{\sigma,-}(\vec{p})$$

$$\Rightarrow \quad \hat{b}_{\sigma}(\vec{p})|vac\rangle = \hat{d}_{\sigma}(\vec{p})|vac\rangle = 0$$

$$\Rightarrow \hat{H} = \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \sum_{\sigma=1,2} p_0 \left[\hat{b}^{\dagger}_{\sigma}(\vec{p}) \hat{b}_{\sigma}(\vec{p}) - \hat{d}_{\sigma}(\vec{p}) \hat{d}^{\dagger}_{\sigma}(\vec{p}) \right]$$

$$\equiv \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \sum_{\sigma=1,2} p_0 \left[\hat{b}^{\dagger}_{\sigma}(\vec{p}) \hat{b}_{\sigma}(\vec{p}) + \hat{d}^{\dagger}_{\sigma}(\vec{p}) \hat{d}_{\sigma}(\vec{p}) \right]$$

normal ordered

$$+ \; E_0 \equiv \; :\hat{H}: + E_0$$

where $E_0 = -2V \int d^3p \sqrt{\vec{p}^2 + m^2} = -2 E_0^{KG}$

and

$$\boxed{\hat{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \sum_{\sigma=1,2} \left[\hat{b}_{\sigma}(\vec{p})\, u^{(\sigma)}(\vec{p})\, e^{-ip\cdot x} + \hat{d}^{\dagger}_{\sigma}(\vec{p})\, v^{(\sigma)}(\vec{p})\, e^{ip\cdot x} \right]}$$

$$\Rightarrow \{ \hat{\psi}_a(\vec{x}), \hat{\psi}^{\dagger}_{a'}(\vec{x}') \} = \delta^3(\vec{x} - \vec{x}')\, \delta_{aa'}$$

(equal-time) $\quad a, a' = 1, \ldots, 4$

$$\{ \hat{\psi}_a(\vec{x}), \hat{\psi}_{a'}(\vec{x}') \} = 0$$

One-Particle States

Let us define the energy momentum 4-vector $\hat{\underline{P}}^\mu$

$$:\hat{\underline{P}}^\mu: \equiv \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} p^\mu \sum_{\sigma=1,2} :\hat{b}^+_\sigma(\vec{p})\, \hat{b}_\sigma(\vec{p}) - \hat{d}_\sigma(\vec{p})\hat{d}^+_\sigma(\vec{p}):$$

$$\equiv \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} p^\mu \sum_{\sigma=1,2} \left[\hat{b}^+_\sigma(\vec{p})\, \hat{b}_\sigma(\vec{p}) + \hat{d}^+_\sigma(\vec{p})\, \hat{d}_\sigma(\vec{p}) \right]$$

$$\Rightarrow :\hat{H}: = :\hat{\underline{P}}^0: = \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} p_0 \sum_{\sigma=1,2} \left(\hat{b}^+_\sigma(\vec{p})\, \hat{b}_\sigma(\vec{p}) + \hat{d}^+_\sigma(\vec{p})\, \hat{d}_\sigma(\vec{p}) \right)$$

and

$$:\vec{\underline{P}}: = \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \vec{p} \sum_{\sigma=1,2}^{2} \left(\hat{b}^+_\sigma(\vec{p})\, \hat{b}_\sigma(\vec{p}) + \hat{d}^+_\sigma(\vec{p})\, \hat{d}_\sigma(\vec{p}) \right)$$

$$\Rightarrow \quad :\hat{H}: \hat{b}^+_\sigma(\vec{p})|vac\rangle = +p_0\, \hat{b}^+_\sigma(\vec{p})|vac\rangle$$

$$:\hat{H}: \hat{d}^+_\sigma(\vec{p})|vac\rangle = +p_0\, \hat{d}^+_\sigma(\vec{p})|vac\rangle$$

$$:\vec{\underline{P}}: \hat{b}^+_\sigma(\vec{p})|vac\rangle = \vec{p}\, \hat{b}^+_\sigma(\vec{p})|vac\rangle$$

$$:\hat{\vec{\underline{P}}}: \hat{d}^+_\sigma(\vec{p})|vac\rangle = \vec{p}\, \hat{d}^+_\sigma(\vec{p})|vac\rangle$$

$\Rightarrow$ these states have the <u>same</u> energy $E = p_0$ and the same momentum $\vec{p}$. What quantum #'s label these 4 states?

Spin

The total angular momentum is

$$\hat{\vec{J}} = \int d^3x \; \hat{\psi}^\dagger(\vec{x}) \left[\frac{\hbar}{i} \vec{x} \wedge \vec{\nabla} + \frac{\hbar}{2} \vec{\Sigma} \right] \hat{\psi}(\vec{x})$$

$$\equiv \hat{\vec{L}} + \hat{\vec{S}}$$

$$\Rightarrow \hat{\vec{S}} = \int d^3x \; \hat{\psi}^\dagger(\vec{x}) \frac{\hbar}{2} \vec{\Sigma} \, \hat{\psi}(\vec{x})$$

where $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \equiv \vec{\sigma}$

In order to measure the spin polarization we go to the rest frame and construct states of the form $\hat{b}_\sigma^\dagger(\vec{0}) |vac\rangle$ and $\hat{d}_\sigma^\dagger(\vec{0}) |vac\rangle$, i.e. $\vec{p} = 0$ states

~~Let Ws to~~ Let $W^\mu = (0, m \vec{\Sigma})$, in the rest frame.

Let $n^\mu = (0, \vec{n})$ be a space like 4-vector in the rest frame s.t. $n^\mu n_\mu = -\vec{n}^2 = -1$

$$\Rightarrow W_\mu n^\mu = -m \, \vec{n} \cdot \vec{\Sigma} \quad \text{in the rest frame.}$$

Since $W_\mu n^\mu$ is a Lorentz scalar, it is equal to $-m\vec{n}\cdot\vec{\Sigma}$ in all Lorentz frames.

$$\Rightarrow \quad W_\mu n^\mu = -\frac{m}{2}\begin{pmatrix} \vec{n}\cdot\vec{\sigma} & 0 \\ 0 & \vec{n}\cdot\vec{\sigma}\end{pmatrix}$$

If $\vec{n} = \hat{e}_z \Rightarrow$

$$W_\mu n^\mu = -\frac{m}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3\end{pmatrix}$$

$$\Rightarrow -\frac{1}{m} W_\mu n^\mu \, u_+^{(1)}(p) = +\frac{1}{2} u_+^{(1)}(p) \qquad \text{spin up}$$

$$-\frac{1}{m} W_\mu n^\mu \, u_+^{(2)}(p) = -\frac{1}{2} u_+^{(2)}(p) \qquad \text{spin down}$$

$$-\frac{1}{m} W_\mu n^\mu \, v_-^{(1)}(p) = +\frac{1}{2} v_-^{(1)}(p)$$

$$-\frac{1}{m} W_\mu n^\mu \, v_-^{(2)}(p) = -\frac{1}{2} v_-^{(2)}(p)$$

It can be shown (after some algebra) that as a consequence of the above identities, the states $\hat{b}_\sigma^\dagger(\vec{0})|vac\rangle$ and $\hat{d}_\sigma^\dagger(\vec{0})|vac\rangle$ have the eigenvalues

$$\hat{S}_z \, \hat{b}_\sigma^\dagger(\vec{0})|vac\rangle = \pm\frac{\hbar}{2} \hat{b}_\sigma^\dagger(\vec{0})|vac\rangle \quad \text{and}$$

$$\hat{S}_z \, \hat{d}_\sigma^\dagger(\vec{0})|vac\rangle = \mp\frac{\hbar}{2} \hat{d}_\sigma^\dagger(\vec{0})|vac\rangle \quad \text{where}$$

the upper sign stands for $\varphi = \binom{1}{0}$ and the lower sign stands for $\varphi = \binom{0}{1}$

$\Rightarrow$ the state $\hat{b}_\sigma^+(\vec{0})|vac\rangle$ has spin $\frac{1}{2}$ with projections $\pm\frac{\hbar}{2}$, whereas the state $\hat{d}_\sigma^+(\vec{0})|vac\rangle$ has spin $\frac{1}{2}$ and projections $\mp\frac{\hbar}{2}$.

Charge and Electromagnetic Coupling

The coupling to an electromagnetic field leads to two extra terms in $\hat{H}$:

$$-\int d^3x \; \hat{\psi}^+(\vec{x})\left[e\,\Phi(\vec{x}) - e\,\vec{\alpha}.\vec{A}(\vec{x})\right]\hat{\psi}(\vec{x}) \equiv$$

$$\equiv -\int d^3x \; \bar{\psi}(\vec{x})\left[e\,\gamma_0\,\Phi(\vec{x}) - e\,\vec{\gamma}.\vec{A}(\vec{x})\right]\hat{\psi}(\vec{x})$$

$$\equiv \int d^3x \; (-e)\; \bar{\psi}(x)\gamma^\mu\psi(x)\; A_\mu(x)$$

$\Rightarrow$ we define a total charge operator.

$$\hat{Q} = -e\int d^3x \; \hat{\psi}^+(\vec{x})\;\hat{\psi}(\vec{x})$$

which is conserved since $[\hat{Q},\hat{H}] = 0$

upon expanding in modes we find

$$\hat{Q} = :\hat{Q}: + Q_{vac}$$

where Q_{vac} is the vacuum charge which must be subtracted off (i.e. we must normal order $\hat{Q}$ so that $Q_{vac} = 0$)

$$\hat{Q} = -e\int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \sum_{\sigma=1,2} \left[\hat{b}^\dagger_\sigma(p)\, \hat{b}_\sigma(p) + \hat{d}_\sigma(p)\, \hat{d}^\dagger_\sigma(p) \right]$$

$$\Rightarrow :\hat{Q}: = -e\int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \sum_{\sigma=1,2} \left[\hat{b}^\dagger_\sigma(p)\, \hat{b}_\sigma(p) - \hat{d}^\dagger_\sigma(p)\, \hat{d}_\sigma(p) \right]$$

$$\Rightarrow :\hat{Q}: |vac\rangle = 0$$

(formally $:\hat{Q}: = -e\int d^3x \, \frac{1}{2} \left[\hat{\psi}^\dagger(\vec{x}), \hat{\psi}(\vec{x}) \right]$)

$$\Rightarrow :\hat{Q}: \hat{b}^\dagger_\sigma(p)|vac\rangle = -e \, \hat{b}^\dagger_\sigma(p)|vac\rangle \qquad \text{negative charge}$$

$$:\hat{Q}: \hat{d}^\dagger_\sigma(p)|vac\rangle = +e \, \hat{d}^\dagger_\sigma(p)|vac\rangle \qquad \text{positive charge}$$

In summary, the Dirac equation (or rather the Dirac Field theory) has a stable ground state provided the fields obey canonical anti-commutation relations.

~~Three are~~ one particle states ~~are~~ have energy $p_0 = \sqrt{\vec{p}^2 + m^2}$
$\equiv \sqrt{\vec{p}^2 c^2 + m^2 c^4}$

(which is always positive) and momentum $\vec{p}$, but for each $\vec{p}$ there are 4 types of one particle states: two states with charge (−e) and spin ±1/2 which we will call the Dirac <u>particle</u> (i.e. the electron) and two states with charge +e and spin ∓1/2 {which we will call the <u>antiparticle</u> (i.e. the positron). All four states have <u>positive</u> excitation energy and are <u>fermions</u>