Relativistic Bound States

We will begin with the case of the Klein - Gordon Equ which applies for particles of spin zero. The KG Equ coupled to an electromagnetic field is

$$\frac{1}{c^2}\left(i\hbar \frac{\partial}{\partial t} - e\Phi\right)^2 \phi(\vec{r},t) - \left[\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 + m^2c^2\right]\phi(\vec{r},t) = 0$$

we will be interested in the case of a central potential

$e\Phi = -\frac{Ze^2}{r}$ and $\vec{B} = 0$. The stationary solutions are

$$\phi(\vec{r},t) = e^{-iEt/\hbar}\,\phi(\vec{r})$$

$$\Rightarrow \left(i\hbar \frac{\partial}{\partial t} - e\Phi\right)\phi(\vec{r},t) = -\left(E + e\Phi\right)e^{-iEt/\hbar}\,\phi(\vec{r})$$

and

$$\Rightarrow \left[\left(E + \frac{Ze^2}{r}\right)^2 + \hbar^2c^2\nabla^2 - m^2c^4\right]\phi(\vec{r}) = 0$$

This equ. has a form quite similar to the non-relativistic Schrödinger equation. The solution has the same form $\phi(\vec{r}) = R(r)\,Y_{lm}(\theta,\varphi)$

where

$$\alpha = \frac{e^2}{\hbar c} \qquad \left[\left(\frac{E^2}{c^2} - m^2c^2\right) + \hbar^2\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{l(l+1) - (Z\alpha)^2}{r^2}\right) + \frac{2Ze^2}{r}\frac{E}{c}\right]R = 0$$

Define: $m' \equiv \frac{E}{c^2}$ and $\ell'(\ell'+1) \equiv \ell(\ell+1) - (Z\alpha)^2$

$$\frac{E^2}{c^2} - m^2c^2 \equiv 2E'm'$$

$$\Rightarrow \quad \left[2m'E' + \hbar^2 \frac{1}{r}\frac{\partial^2}{\partial r^2} r - \frac{\ell'(\ell'+1)}{r^2} + \frac{2m' Ze^2}{r}\right] R(r) = 0$$

which is the Radial Schrödinger Eqn. The only difference is that $\ell' \notin \mathbb{Z}$!

$$\ell' = \ell - \delta_\ell \Rightarrow \quad (\ell - \delta_\ell)(\ell+1-\delta_\ell) = \ell(\ell+1) - (Z\alpha)^2$$

$$\ell\cancel{(\ell+1)} - (2\ell+1)\delta_\ell + \delta_\ell^2 = \ell\cancel{(\ell+1)} - (Z\alpha)^2$$

$$\delta_\ell^2 - (2\ell+1)\delta_\ell + (Z\alpha)^2 = 0$$

$$\delta_\ell = \left(\ell+\tfrac{1}{2}\right) - \sqrt{\left(\ell+\tfrac{1}{2}\right)^2 - (Z\alpha)^2}$$

$$\Rightarrow \quad \ell' = \ell - \delta_\ell = -\frac{1}{2} + \sqrt{\left(\ell+\tfrac{1}{2}\right)^2 - (Z\alpha)^2}$$

Similarly, n is displaced by the same amount since $n' = n - (\ell+1) \in \mathbb{Z}$

$\Rightarrow$ the eigenvalues are

$$\frac{E_{n\ell}^2 - m^2c^4}{2mc^2} = -\frac{mc^2 Z^2\alpha^2}{2} \frac{E_{n\ell}^2}{m^2c^4} \frac{1}{(n-\delta_\ell)^2}$$

$$\Rightarrow \quad E_{n\ell} = \frac{mc^2}{\sqrt{1 + \frac{Z^2\alpha^2}{(n-\delta_\ell)^2}}}$$

Comments

① the ~~ei~~ eigenvalues depend on both $\underline{n}$ and $\underline{\ell}$ but not $\underline{\underline{m}}$ (as expected)

② the $l=0$ (S wave) state is not allowed for $Z > \frac{1}{\alpha} \simeq 137$

The Dirac Case

The Dirac bound state problem can be solved using similar ideas. The main difference is the role of spin. To simplify the analysis I will use relativistic units: $\hbar = c = 1$. The Dirac equation is $\left(P_\mu = i\partial_\mu = i \frac{\partial}{\partial x^\mu} \right)$

$$\left[\gamma^\mu (P_\mu + e A_\mu) - m\right] \psi(x) = 0$$

Multiply from the left by $\gamma^\nu (P_\nu + e A_\nu) + m$

$$\Rightarrow \left[\gamma^\mu \gamma^\nu (P_\mu + eA_\mu)(P_\nu + eA_\nu) - m^2\right] \psi = 0$$

$$\Sigma^{\mu\nu} \equiv \frac{i}{2} \left[\gamma^\mu, \gamma^\nu\right]$$

$$\Sigma^{0i} = i\,\alpha_i \qquad \text{and} \qquad \Sigma^{ij} = \varepsilon_{ijk}\, \Sigma_k$$

$$\Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} = 2 S_i$$

$$\gamma_\mu \gamma_\nu = \tfrac{1}{2}\{\gamma_\mu, \gamma_\nu\} + \tfrac{1}{2}[\gamma_\mu, \gamma_\nu]$$

$$\Rightarrow \gamma^\mu \gamma^\nu = g^{\mu\nu} - i\,\Sigma^{\mu\nu}$$

$$\Rightarrow \left[(p^\mu + eA^\mu)(p_\mu + eA_\mu) - i\,\Sigma^{\mu\nu}(p_\mu + eA_\mu)(p_\nu + eA_\nu) - m^2\right]\psi = 0$$

$$-i\,\Sigma^{\mu\nu}(i\partial_\mu + eA_\mu)(i\partial_\nu + eA_\nu)\,\psi =$$

$$= -e\,\Sigma^{\mu\nu}\,\partial_\mu A_\nu\,\psi$$

$$= -\frac{e}{2}\,\Sigma^{\mu\nu} F_{\mu\nu}\,\psi = -e\left(-\vec{\Sigma}\cdot\vec{B} + i\vec{\alpha}\cdot\vec{E}\right)\psi$$

$$\Rightarrow \left[(E + eA^0)^2 - (\vec{p} + e\vec{A})^2 + e\,\vec{\Sigma}\cdot\vec{B} - ie\vec{\alpha}\cdot\vec{E} - m^2\right]\psi = 0$$

$$eA^0 = \frac{\alpha}{r}Z \quad , \quad \vec{A} = 0$$

$$\cancel{\left(E + \frac{Z\alpha}{r}\right)^2 + p^2}$$

$$\left[\left(E + \frac{Z\alpha}{r}\right)^2 - p^2 - ie\vec{\alpha}\cdot\vec{E} - m^2\right]\psi = 0$$

$$\Rightarrow \left[\nabla^2 + \frac{2EZ\alpha}{r} + \frac{(Z\alpha)^2}{r^2} + E^2 - iZ\alpha\,\frac{\vec{\alpha}\cdot\hat{r}}{r^2} - m^2\right]\psi = 0$$

$$\left[\frac{1}{2mr^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{\left(L^2 - (Z\alpha)^2\right)}{2mr^2} + \frac{EZ\alpha}{mr} - i\alpha Z\,\frac{\vec{\alpha}\cdot\hat{r}}{r^2} + \frac{E^2 - m^2}{2m}\right]\psi = 0$$

We will look at positive energy states and choose a basis of eigenstates of $\vec{J}^2$, J_z and $\vec{L}^2$ (even though these are not eigenstates of H).

For each value of j and m there are two eigenstates of $\vec{L}^2$, $l = j \pm \frac{1}{2}$

$[L^2, \vec{\alpha}.\hat{r}] \neq 0$ and $\vec{\alpha}.\hat{r}$ $l = j \pm \frac{1}{2} \to j \mp \frac{1}{2}$

The eigenstates are linear combinations.

$(\vec{\alpha}.\hat{r})^2 = 1 \Rightarrow$ the off-diagonal value is ± 1

radial equation:

$$\left(\frac{1}{2mr^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{1}{2mr^2}\underset{\text{matrix}}{M} + \frac{E Z\alpha}{mr} - \frac{E^2 - m^2}{2m}\right)\psi = 0$$

$$M = \begin{pmatrix} \vec{L}^2 - (Z\alpha)^2 & \pm i Z\alpha \\ \pm i Z\alpha & \vec{L}^2 - (Z\alpha)^2 \end{pmatrix}$$

$$= \begin{pmatrix} (j+\frac{1}{2})(j+\frac{3}{2}) - (Z\alpha)^2 & \pm i Z\alpha \\ \pm i Z\alpha & j^2 - \frac{1}{4} - (Z\alpha)^2 \end{pmatrix}$$

$\lambda_1 = \mu(\mu+1)$ $\lambda_2 = \mu(\mu-1)$ e.v.'s

$$\mu = \sqrt{(j+\tfrac{1}{2})^2 - (Z\alpha)^2}$$

The only normalizable solutions are

$$E = \frac{m}{\sqrt{1 + \frac{(Z\alpha)^2}{(n'+l'+1)^2}}}$$

where $l' = \mu$ or $\mu - 1$ and $n' \geq 0$

$$\Rightarrow E = \frac{mc^2}{\sqrt{1 + \frac{(Z\alpha)^2}{n - j - \frac{1}{2} + \sqrt{(j+\frac{1}{2})^2 - (Z\alpha)^2}}}}$$

$n = 1, 2, \ldots$

$j = \frac{1}{2}, \frac{3}{2}, \ldots, n - \frac{1}{2}$

is the spectrum.

There are two combination of n' and l' for each n and j <u>except</u> for the highest allowed value of j for each n.