Relativistic Quantum Mechanics

Except for our discussion of the quantized electromagnetic field, we have ignored until now the ~~effects~~ question of how to make Quantum Mechanics compatible with the Theory of Special Relativity. The basic problem is that the Schrödinger Equation

$$i\hbar \frac{\partial \psi}{\partial t} = H \psi$$

is not relativistically covariant, i.e. it changes form under Lorentz Transformations. This is so for a number of reasons. One is that in the way we quantized these systems we used a procedure called canonical quantization which makes use of equal-time commutation relations. Thus, time and space a-priori play a different role. While it is possible to find ways of reconciling this approach with the ~~g~~ requirement of covariance, it is not a-priori guaranteed to hold. For instance, you may think that the problem is the use of a non-relativistic Hamiltonian. Thus one possibility is to replace

$$\frac{\hat{\vec{p}}^2}{2m} \longrightarrow \sqrt{\hat{\vec{p}}^2 c^2 + m^2 c^4} \qquad \text{with } \vec{P} = \frac{\hbar}{i}\vec{\nabla}$$

(as usual)

However, although this agrees with the classical (i.e. not quantum) relativistic dispersion it is ~~still~~ not good for a # of reasons,

(1) The square root means that it is non-local (i.e. all powers of ∇^2 show up in the expansion)

(2) It is not relativistically covariant since space and time enter asymmetrically

(3) It turns out to violate <u>Causality</u>

In fact, the Schrödinger Equation now is

$$i\hbar \frac{\partial \psi}{\partial t}(\vec{x},t) = \int d^3x' \, K(\vec{x}-\vec{x}') \, \psi(\vec{x}',t)$$

where $$K(\vec{x}-\vec{x}') = \int \frac{d^3p}{(2\pi\hbar)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')/\hbar} \sqrt{\vec{p}^2c^2+m^2c^4}$$

which is important for $|\vec{x}-\vec{x}'| \lesssim \lambda_c = \frac{\hbar}{mc}$ (Compton wave length)

$\Rightarrow$ the way $\psi(\vec{x},t)$ changes in time depends on the values of $\psi(\vec{x}',t)$ <u>within the Compton wavelength</u> but <u>outside the forward light cone</u> centered at $\vec{x}$ $\Rightarrow$ <u>violation of causality</u>

This form of the Schrödinger Equation is thus inconsistent. However, the quantity

$\frac{E^2}{c^2} - p^2c^2 = m^2c^2$ is <u>Lorentz invariant</u>

$\Rightarrow$ we can write instead the Klein-Gordon Equation

$$\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t}\right)^2\psi = \left(\frac{\hbar}{i}\vec{\nabla}\right)^2\psi + m^2c^2\psi$$

$$\Rightarrow \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right]\psi(\vec{x},t) = 0$$

which looks like a classical wave equation (2nd. order in $\frac{\partial}{\partial t}$!) with the extra term

$$\sim \left(\frac{mc}{\hbar}\right)^2 = \frac{1}{\lambda^2_{\text{Compton}}}$$

(1) This equation, is, by construction, relativistically <u>invariant</u> and we can write it in the compact

form

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right) \psi(\vec{x},t) = 0$$

where $\quad \Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$, as before

(2) It is coupled to an electromagnetic field with scalar potential ϕ and vector potential $\vec{A}$ by the ~~substitution~~ substitution

$$i\hbar\frac{\partial}{\partial t} \longrightarrow i\hbar\frac{\partial}{\partial t} - e\phi(\vec{x},t)$$

and

$$\frac{\hbar}{i}\vec{\nabla} \longrightarrow \frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}(\vec{x},t)$$

$$\Rightarrow \boxed{\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t} - e\phi(\vec{x},t)\right)^2 \psi(\vec{x},t) = \left[\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}(\vec{x},t)\right)^2 + \frac{m^2c^2}{\hbar^2}\right]\psi}$$

(3) Since $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ has the same form in all Lorentz frames, $\Rightarrow$ Lorentz invariance requires that under a Lorentz transformation

$$\psi'(\vec{x}',t') = \psi(\vec{x},t) \Rightarrow \psi \text{ is a Lorentz scalar}$$

$$x' = \Lambda x$$

Λ: Lorentz transformation

Note: there is no spin!

(4) However, the price we paid is that the KG equation is <u>second order in time derivatives</u> $\Rightarrow$ we must now specify both $\psi(\vec{x},t_0)$ <u>and</u> $\frac{\partial}{\partial t}\psi(\vec{x},t_0) \Rightarrow \exists$ more degrees of freedom $\Rightarrow$ antiparticles (see below)

(5) free particle solutions

$$\psi(\vec{x},t) \propto e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - Et)}$$

$$\Rightarrow \quad E^2 = \vec{p}^2c^2 + m^2c^4 \quad \checkmark$$

but $\quad E = \pm\sqrt{p^2c^2 + m^2c^4} \quad$ negative energies?

(6) Is there a conserved probability current?

$$\psi^*\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right]\psi - \psi\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right]\psi^* = 0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

with $\quad \vec{J}(\vec{x},t) = \frac{\hbar}{2mi}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$

and $\quad \rho(\vec{x},t) = \frac{i\hbar}{2mc^2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)$

$$\Rightarrow \quad \frac{\partial}{\partial t}\int d^3x \; \rho(\vec{x},t) = 0 \Rightarrow \text{conservation}$$

but $\rho(\vec{x},t)$ may be positive or negative!

In fact, in the presence of an e.m. field

$$\vec{J}(\vec{x},t) = \frac{1}{2m}\left[\psi^*\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}\right)\psi + \psi\left(-\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}\right)\psi^*\right]$$

and

$$\rho(\vec{x},t) = \frac{1}{2mc^2}\left[\psi^*\left(i\hbar\frac{\partial}{\partial t} - e\phi\right)\psi + \psi\left(-i\hbar\frac{\partial}{\partial t} - e\phi\right)\psi^*\right]$$

with

$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$ still holds $\Rightarrow$ charge is conserved.

$\Rightarrow$ $e\vec{J}(\vec{x},t)$ is the charge current and

$e\rho(\vec{x},t)$ is the charge density.

$\Rightarrow$ If $e\rho < 0 \Rightarrow$ negative charge

$e\rho > 0 \Rightarrow$ positive charge

We will see that there is a consistent interpretation in terms of the excitations of a <u>field</u>, but not as a <u>wave function</u>

We will now see that these inconsistencies are resolved if we regard $\psi(\vec{x},t)$ not as a <u>wave function</u> but as a <u>field</u> whose excitations are the particles, just as photons are the excitations of the quantized electromagnetic field. This point of view implies that we must abandon the picture of Quantum Mechanics with a fixed number of particles and that we have replaced the Hilbert space by a Fock space. Thus, much as we did before, we will quantize this <u>field</u> by imposing equal-time commutation-relations between the field ϕ (we switch to call it ϕ for historical reasons) and its <u>canonical momentum</u> $\Pi = \frac{\partial \phi}{\partial t}$. Since we are working with a relativistic system we will chose units s.t. $c=1$. Also we will choose units s.t. $\hbar = 1 \Rightarrow [E] = \frac{1}{[T]}$ and $c = 1 \Rightarrow$ $[L] = [T]$

$$\Rightarrow \quad [\hat{\phi}(\vec{x}), \hat{\Pi}(\vec{y})] = i\,\delta^3(\vec{x}-\vec{y})$$

but $[\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})] = [\hat{\Pi}(\vec{x}), \hat{\Pi}(\vec{y})] = 0$

We will construct a quantum Hamiltonian s.t. the equation of motion of $\hat{\phi}(\vec{x},t)$ is the Klein-Gordon Equation $(\Box + m^2)\hat{\phi}(\vec{x},t) = 0$

$$\Box \equiv \frac{\partial^2}{\partial t^2} - \nabla^2 \qquad (c=1)$$

$$m = \frac{1}{\lambda_{\text{compton}}} \qquad (\hbar = 1)$$

It is easy to see that the Hamiltonian is

$$\hat{H} = \int d^3x \left\{ \frac{1}{2}\hat{\Pi}^2(\vec{x}) + \frac{1}{2}(\vec{\nabla}\hat{\phi})^2 + \frac{1}{2}m^2\hat{\phi}^2 \right\}$$

Check:

$$i\frac{\partial\hat{\phi}}{\partial t} = [\hat{\phi}, \hat{H}] = i\hat{\Pi} \Rightarrow \hat{\Pi} = \frac{\partial\hat{\phi}}{\partial t}$$

$$i\frac{\partial\hat{\Pi}}{\partial t} = [\hat{\Pi}, \hat{H}] \Rightarrow$$

$$\Rightarrow \quad i\,\partial_t^2\hat{\phi} = i\,[\nabla^2\hat{\phi} - m^2\hat{\phi}]$$

$$\Rightarrow \quad (\partial_t^2 - \nabla^2)\hat{\phi} + m^2\hat{\phi} = 0$$

$$\Leftrightarrow \quad (\Box + m^2)\hat{\phi} = 0 \qquad \checkmark$$

Let us expand the field in plane waves; i.e. Fourier

$$\hat{\phi}(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} \, e^{i\vec{p}\cdot\vec{x}} \, \hat{\phi}(\vec{p},t)$$

$$KG \Rightarrow \quad \left[\frac{\partial^2}{\partial t^2} + (m^2 + \vec{p}^2)\right] \hat{\phi}(\vec{p},t) = 0$$

$$\Rightarrow \quad \hat{\phi}(\vec{p},t) = \text{const} \; e^{\pm i\omega(\vec{p})t}$$

$$\omega(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$$

Since we have two solutions ($\pm$) the mode expansion is

$$\hat{\phi}(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{p})}} \left[e^{i\vec{p}\cdot\vec{x}} \, \hat{a}(\vec{p},t) + e^{-i\vec{p}\cdot\vec{x}} \, \hat{a}^\dagger(\vec{p},t) \right]$$

convention

$$\hat{\Pi}(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega(\vec{p})}{2}} \left[\hat{a}(\vec{p},t)\, e^{i\vec{p}\cdot\vec{x}} - \hat{a}^\dagger(\vec{p},t)\, e^{-i\vec{p}\cdot\vec{x}} \right]$$

or, at $t=0$

$$\hat{\phi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{p})}} \, e^{i\vec{p}\cdot\vec{x}} \left(\hat{a}(\vec{p}) + \hat{a}^\dagger(-\vec{p}) \right)$$

$$\hat{\Pi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega(\vec{p})}{2}} \, e^{i\vec{p}\cdot\vec{x}} \left(\hat{a}(\vec{p}) - \hat{a}^\dagger(-\vec{p}) \right)$$

$$\Rightarrow \text{if } \left[\hat{a}(\vec{p}), \hat{a}^{+}(\vec{p}\,')\right] = (2\pi)^3 \delta^3(\vec{p}-\vec{p}\,')$$

$$\Rightarrow \left[\hat{\phi}(\vec{x}), \hat{\Pi}(\vec{x}\,')\right] = i\,\delta^3(\vec{x}-\vec{x}\,') \quad \checkmark$$

$$\text{and } \left[\hat{a}(\vec{p}), \hat{a}(\vec{p}\,')\right] = \left[\hat{a}^{+}(\vec{p}), \hat{a}^{+}(\vec{p}\,')\right] = 0$$

$$\Rightarrow \left[\hat{\phi}(\vec{x}), \hat{\phi}(\vec{x}\,')\right] = \left[\hat{\Pi}(\vec{x}), \hat{\Pi}(\vec{x}\,')\right] = 0$$

$$\Rightarrow \hat{H} = \int d^3x \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3}\, e^{i(\vec{p}+\vec{p}\,')\cdot\vec{x}}$$

$$\Big\{ - \frac{\sqrt{\omega(\vec{p})\,\omega(\vec{p}\,')}}{4} \left(\hat{a}(\vec{p}) - \hat{a}^{+}(-\vec{p})\right)\left(\hat{a}(\vec{p}\,') - \hat{a}^{+}(-\vec{p}\,')\right)$$

$$+ \frac{-\vec{p}\cdot\vec{p}\,' + m^2}{4\sqrt{\omega(\vec{p})\,\omega(\vec{p}\,')}} \left(\hat{a}(\vec{p}) + \hat{a}^{+}(-\vec{p})\right)\left(\hat{a}(\vec{p}\,') + \hat{a}^{+}(-\vec{p}\,')\right)$$

$$\Rightarrow H = \int \frac{d^3p}{(2\pi)^3}\, \omega(\vec{p})\, \frac{1}{2}\left[\hat{a}^{+}(\vec{p})\hat{a}(\vec{p}) + \hat{a}(\vec{p})\hat{a}^{+}(\vec{p})\right]$$

$$\Rightarrow H = \int \frac{d^3p}{(2\pi)^3}\, \omega(\vec{p})\, \hat{a}^{+}(\vec{p})\,\hat{a}(\vec{p}) + \text{zero point energy}$$

(compare with the theory of photons)

What is the total linear momentum? (i.e. the generator of translations)

$$\vec{\mathbb{P}} = \int \frac{d^3p}{(2\pi)^3} \; \vec{p} \; \hat{a}^{\dagger}(\vec{p})\, \hat{a}(\vec{p}) \equiv -\int d^3x \; \hat{\Pi}(\vec{x}) \vec{\nabla} \hat{\phi}(\vec{x})$$

Vacuum state: $|0\rangle$

$$\hat{a}(\vec{p})\,|0\rangle = 0$$

$$\Rightarrow \hat{\vec{\mathbb{P}}}\,|0\rangle = 0 \quad \text{and} \quad \hat{H}\,|0\rangle = E_0 |0\rangle$$

$$E_0 = V \int d^3p \; \frac{\omega(\vec{p})}{2}$$

which is divergent (just as in Maxwell's)

Notice that all the energies are positive; the field has positive and negative frequency components associated with the creation and destruction operators of its excitations

$\Rightarrow$ antiparticles are negative frequency components but positive energy states.

Particles: $|\vec{p}\rangle \equiv \sqrt{2E(\vec{p})}\ \hat{a}^{\dagger}(\vec{p})|0\rangle$

$$E(\vec{p}) = +\sqrt{p^2+m^2}$$

$\Rightarrow \langle \vec{p}|\vec{q}\rangle = 2E(\vec{p})(2\pi)^3\delta^3(\vec{p}-\vec{q})$ which is a Lorentz scalar.

$\Rightarrow \hat{H}|\vec{p}\rangle = E(\vec{p})|\vec{p}\rangle$

and $\hat{\vec{P}}|\vec{p}\rangle = \vec{p}\ |\vec{p}\rangle$

$\Rightarrow$ these are the single particle states

What statistics do these particles obey?

Consider a two-particle state

$$|\vec{p},\vec{p}\,'\rangle = \sqrt{2E(\vec{p})}\sqrt{2E(\vec{p}\,')}\ \hat{a}^{\dagger}(\vec{p})\,\hat{a}^{\dagger}(\vec{p}\,')|0\rangle$$

Since $[\hat{a}^{\dagger}(\vec{p}), \hat{a}^{\dagger}(\vec{p}\,')] = 0$ this is a <u>symmetric</u> state $\Rightarrow$ these excitations (particles) are <u>bosons</u> and have <u>spin zero</u>

(since $\hat{\phi}(\vec{x},t) = \hat{\phi}'(\vec{x}\,',t')$)

$\Rightarrow$ The Klein-Gordon field describes particles such as pions, Kaons, etc. but <u>not</u> electrons protons, neutrons, neutrinos, which carry spin 1/2 and are <u>fermions</u> $\Rightarrow$ Dirac Equation.