Identical Particles at Finite Density and Temperature

Here we will consider a system of identical particles at finite density, $\rho = \frac{N}{V} < \infty$, and at $T > 0$. For simplicity we will only discuss the free particle non-relativistic case, i.e. a quantum ideal gas. We will consider the case of both bosons and fermions.

(A) Fermions

The (grand) partition function at fixed T and at fixed chemical potential μ is

$$Z(\mu, V, T) = \mathrm{Tr}\, e^{-\beta(H - \mu N)}$$

$$= \sum_{\{n_\alpha\}} e^{-\beta \sum_\alpha (E_\alpha - \mu) n_\alpha}$$

$$N = \sum_\alpha a^\dagger_\alpha a_\alpha \quad ; \quad H = \sum_\alpha E_\alpha a^\dagger_\alpha a_\alpha$$

Here α label a complete set of one-particle states.
For the case of fermions, $n_\alpha = 0, 1$

$$\Rightarrow \quad Z = \prod_\alpha \sum_{n_\alpha = 0,1} e^{-\beta n_\alpha (E_\alpha - \mu)} = \prod_\alpha \left(1 + e^{-\beta(E_\alpha - \mu)}\right)$$

(Gibbs)

The Free energy is

$$F = -kT \log Z$$

$$= -kT \sum_\alpha \log\left(1 + e^{-\beta(E_\alpha - \mu)}\right)$$

What is the average occupation number $\langle n_\alpha \rangle$ of a single particle state α?

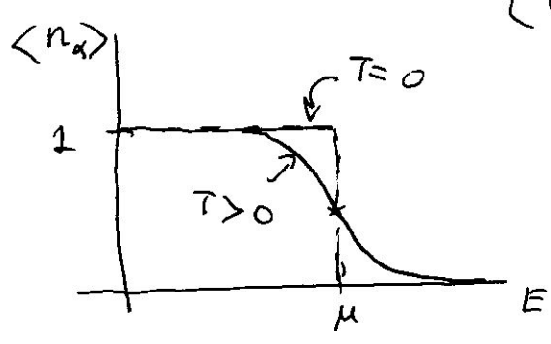
$$\langle n_\alpha \rangle = \frac{\sum_{n_\alpha = 0,1} n_\alpha \, e^{-\beta n_\alpha (E_\alpha - \mu)}}{\sum_{n_\alpha = 0,1} e^{-\beta n_\alpha (E_\alpha - \mu)}}$$

$$= \frac{0 + e^{-\beta(E_\alpha - \mu)}}{1 + e^{-\beta(E_\alpha - \mu)}}$$

$$\Rightarrow \langle n_\alpha \rangle = \frac{1}{e^{\beta(E_\alpha - \mu)} + 1}$$ Fermi-Dirac distribution.

For $T \to 0$ $(\beta \to \infty)$ $\quad \langle n_\alpha \rangle = \begin{cases} 1 & E_\alpha < \mu \\ 0 & E_\alpha > \mu \end{cases}$

$\Rightarrow \mu = E_F$



On the other hand $N = \langle N \rangle = \sum_\alpha \langle n_\alpha \rangle$

$$\Rightarrow \quad N = \sum_\alpha \frac{1}{e^{\beta(E_\alpha - \mu)} + 1}$$

This equ. determines $\mu = \mu(N, T)$

For a free non-relativistic $(s=1/2)$ Fermi gas at $T=0$

we have $|\alpha\rangle = |\vec{p}\rangle$

$$N = 2V \int_{|\vec{p}| \leq p_F} \frac{d^3p}{(2\pi\hbar)^3} \, 1 = \frac{2V}{(2\pi\hbar)^3} \frac{4\pi}{3} p_F^3$$

$$\Rightarrow \quad \rho = \frac{N}{V} = \frac{1}{3\pi^2} \left(\frac{p_F}{\hbar}\right)^3 \Longleftrightarrow p_F = \hbar \left(3\pi^2 \rho\right)^{1/3}$$

The Fermi energy is $E_F = \mu = \dfrac{p_F^2}{2m} = \dfrac{\hbar^2}{2m} \left(3\pi^2 \rho\right)^{2/3}$

The total energy E is

$$E = 2V \int_{|\vec{p}| \leq p_F} \frac{d^3p}{(2\pi\hbar)^3} \frac{p^2}{2m}$$

$$\frac{E}{V} = \frac{2}{(2\pi\hbar)^3 \, 2m} \frac{4\pi}{5} p_F^5 = \frac{\hbar^2}{10\pi^2 m} \left(\frac{p_F}{\hbar}\right)^5 = \frac{\hbar^2}{10\pi^2 m} \left(3\pi^2 \rho\right)^{5/3}$$

$$E = \frac{\hbar^2}{10\pi^2 m}\left(3\pi^2 N\right)^{5/3} V^{-2/3}$$

The pressure at $T=0$ is then $\neq 0$!

$$P_0 = -\left.\frac{\partial E}{\partial V}\right|_N = \frac{2}{3}\left(\frac{E}{V}\right)$$

Fermi Pressure

$$P_0 = \frac{2}{3}\left(\frac{E}{V}\right) = \frac{\hbar^2}{15\pi^2 m}\left(\frac{p_F}{\hbar}\right)^5 = \frac{\hbar^2}{15\pi^2 m}\left(3\pi^2\rho\right)^{5/3}$$

This result is a consequence of the Pauli Principle and reflects the fact that fermions avoid each other. Since there is a non-zero pressure one may ask what is exerting a force. The answer, for a free particle system, is that the walls of the container (or the confining potential which is the same) are providing the requisite force.

Collapsed Stars

There is however one case in which an external force is not needed. This is what happens in a neutron star. In this case m is the mass of a neutron and the extra force is the attractive gravitational force of all the neutrons combined. If we think of the star as a uniform fluid of neutrons of mass M and radius R, the gravitational potential energy can be estimated from Newton's Law of Universal Gravitation, to be

$$W = -\frac{3}{5} G M^2 \left(\frac{4\pi}{3V}\right)^{1/3}$$

and the gravitational pressure is

$$P_G = \frac{dW}{dV} = \frac{3}{4\pi} \frac{G M^2}{5 R^4}$$

Hydrostatic equilibrium $\Rightarrow$ $P_G = P_0$

$$\Rightarrow \frac{3}{4\pi} \frac{G M^2}{5 R^4} = \frac{\hbar^2}{15\pi^2 m} \left(3\pi^2 \rho\right)^{5/3} = \frac{\hbar^2}{15\pi^2 m} \left(3\pi^2\right)^{5/3} \left(\frac{M}{m \frac{4\pi R^3}{3}}\right)^{5/3}$$

$$\Rightarrow R = \frac{\hbar^2}{G m M^2} \left(\frac{9\pi}{4}\right)^{2/3} \left(\frac{M}{m}\right)^{5/3}$$ is the equilibrium radius

For a large star the Fermi energy becomes so large that relativistic effects become important. This happens whenever

$$E_F \simeq mc^2 \quad \text{or} \quad p_F \simeq mc$$

$$\Rightarrow \quad \hbar\left(3\pi^2 \rho_c\right)^{1/3} \approx mc \quad \Leftrightarrow \quad \rho_c \approx \left(\frac{mc}{\hbar}\right)^3 \frac{1}{3\pi^2} = \frac{1}{3\pi^2} \frac{1}{\left(\frac{\hbar}{mc}\right)^3}$$

$$\hbar\left(3\pi^2 \frac{M}{\frac{4\pi}{3} R^3 m}\right)^{1/3} \approx mc$$

or $$\frac{\hbar}{mc}\left(\frac{9\pi}{4}\right)^{1/3}\left(\frac{M}{m}\right)^{1/3} \approx R$$

$$\Rightarrow \quad \frac{\hbar c}{Gm^2} \approx \left(\frac{4}{9\pi}\right)^{1/3}\left(\frac{M}{m}\right)^{1/3}$$

Planck mass: $\sqrt{\frac{\hbar c}{G}} = m_{Pl} \sim 2 \times 10^{-5}\,g$

In this regime our non-relativistic formulas don't work and we need to treat the fluid as a gas of relativistic particles. Their energy now is

$$E(p) = \sqrt{m^2c^4 + p^2c^2}$$

the total energy now is (at $T=0$)

$$E = \frac{Vc}{\pi^2\hbar^3} \int_0^{p_F} dp \; p^2 \sqrt{p^2 + m^2c^2}$$

The relation between p_F and ρ remains the same as before

$$p_F = \hbar \left(3\pi^2\rho\right)^{1/3}$$

$$E = \frac{(mc)^4 Vc}{\pi^2\hbar^3} \int_0^{\frac{p_F}{mc}} dx \; x^2\sqrt{1+x^2}$$

$$= \frac{(mc)^4 Vc}{8\pi^2\hbar^3} \left\{ \left(1 + 2\left(\frac{p_F}{mc}\right)^2\right) \frac{p_F}{mc} \sqrt{1+\left(\frac{p_F}{mc}\right)^2} - \sinh^{-1}\left(\frac{p_F}{mc}\right) \right\}$$

$$\sinh^{-1}\left(\frac{p_F}{mc}\right) = \ln\left[\frac{p_F}{mc} + \sqrt{1+\left(\frac{p_F}{mc}\right)^2}\right]$$

We are interested in the high density limit

$p_F \gg mc$

$$\Rightarrow \quad E = \frac{\hbar c}{4\pi^2}\left(3\pi^2\frac{M}{m}\right)^{\frac{4}{3}} V^{-\frac{1}{3}} + \frac{\hbar c}{4\pi^2}\left(3\pi^2\frac{M}{m}\right)^{2/3}\left(\frac{mc}{\hbar}\right)^2 V^{\frac{1}{3}} + \dots$$

and

$$P_0 = -\frac{\partial E}{\partial V} = \frac{\hbar c}{12\pi^2}\left[\left(3\pi^2 \frac{M}{m}\frac{1}{V}\right)^{\frac{4}{3}} - \left(\frac{mc}{\hbar}\right)^2 \left(3\pi^2 \frac{M}{m}\frac{1}{V}\right)^{2/3} + \cdots\right]$$

$$= \frac{\hbar c}{12\pi^2}\left\{\left(\frac{9\pi}{4}\frac{M}{m}\right)^{\frac{4}{3}}\frac{1}{R^4} - \left(\frac{9\pi}{4}\frac{M}{m}\right)^{2/3}\frac{(mc/\hbar)^2}{R^2} + \cdots\right\}$$

To achieve hydrostatic equilibrium we must still have

$$P_0 = P_G$$

$$\Rightarrow \frac{\hbar c}{12\pi^2}\left(\frac{9\pi}{4}\frac{M}{m}\right)^{\frac{4}{3}}\frac{1}{R^4} - \frac{\hbar c}{12\pi^2}\left(\frac{9\pi}{4}\frac{M}{m}\right)^{\frac{2}{3}}\frac{(mc)^2}{\hbar^2 R^2} =$$

$$= \frac{3}{4\pi}\frac{GM^2}{5R^4}$$

Notice that both sides have a $\frac{1}{R^4}$ behavior.

$$\Rightarrow \left(\frac{\hbar c}{12\pi^2}\left(\frac{9\pi}{4}\frac{M}{m}\right)^{\frac{4}{3}} - \frac{3}{20\pi}GM^2\right)\frac{1}{R^4} \simeq \frac{\hbar c}{12\pi^2}\left(\frac{9\pi}{4}\frac{M}{m}\right)^{2/3}\frac{(mc)^2}{\hbar^2 R^2} + \cdots$$

which has a solution only if

$$\frac{\hbar c}{12\pi^2}\left(\frac{9\pi}{4}\frac{M}{m}\right)^{4/3} \geq \frac{3}{20\pi}GM^2$$

Or, what is the same,

$$M \leq M_c = m \left(\frac{M_{pl}}{m}\right)^3 \left(\frac{5}{4}\right)^{3/2} \left(\frac{9\pi}{4}\right)^{1/2}$$

where $M_{pl} = \sqrt{\frac{\hbar c}{G}}$ is the Planck mass.

$\Rightarrow$ $\exists$ a maximum mass M_c. If $M > M_c$ the neutron star will collapse forever and become a black hole! (Chandrasekhar Limit)

For $M \leq M_c$,

$$R = \frac{\hbar}{mc} \left(\frac{9\pi}{4} \frac{M}{m}\right)^{\frac{1}{3}} \left(1 - \left(\frac{M}{M_c}\right)^{2/3}\right)^{1/2}$$

i.e. as $M \to M_c$, $R \to 0$

How large is p_F?

$$p_F = \hbar \left(3\pi^2 \rho\right)^{1/3} = \hbar \left(3\pi^2 \frac{M}{m} \frac{1}{V}\right)^{1/3}$$

$$p_F = \hbar \left(\frac{9\pi}{4} \frac{M}{m}\right)^{1/3} \frac{1}{R}$$

$$\Rightarrow p_F = \frac{mc}{\left(1 - \left(\frac{M}{M_c}\right)^{2/3}\right)^{1/2}}$$

$\Rightarrow$ Clearly as $M \to M_c$, $p \gg mc$ and we need General Relativity corrections.

(B) Bosons

The ground state of a system of (N) free bosons

$$H = \sum_{\alpha} E_\alpha a^\dagger_\alpha a_\alpha$$

is $|gnd\rangle = \cancel{} (a_0^\dagger)^N |0\rangle$

we put all bosons in the same single-particle state $\alpha = 0$

In momentum space (spinless bosons)

$$H = V \int \frac{d^3p}{(2\pi\hbar)^3} \left(\frac{p^2}{2m} - \mu\right) a^\dagger(p)\, a(p)$$

$$|gnd\rangle = \frac{1}{\sqrt{N!}} a^\dagger(\vec{p}=0)^N |0\rangle$$

Bose - Einstein Condensate

$\mu = 0$

$\langle gnd|\, a^\dagger(\vec{x})\, a(\vec{y})\, |gnd\rangle =$

$$= \int \frac{d^3p}{(2\pi\hbar)^3} \int \frac{d^3q}{(2\pi\hbar)^3}\; e^{-i\vec{p}\cdot\vec{x}/\hbar}\, e^{i\vec{q}\cdot\vec{y}/\hbar}$$

$$\langle gnd|\, a^\dagger(\vec{p})\, a(\vec{q})\, |gnd\rangle$$

Using that

$$a(q)\,(a^+(0))^N = N\,(2\pi\hbar)^3\delta^3(q)\,(a^+(0))^{N-1}$$

$$+\,(a^+(0))^N a(q)$$

and that $a(q)\,(a^+(0))^N|0\rangle =$

$$= N\,(2\pi\hbar)^3\,\delta^3(q)\,(a^+(0))^{N-1}|0\rangle$$

and, similarly

$$\langle 0|(a(0))^N\,a^+(p) = N\,(2\pi\hbar)^3\,\delta^3(p)\,\langle 0|(a(0))^{N-1}$$

we find

$$\langle gnd|\,a^+(x)\,a(y)\,|gnd\rangle =$$

$$= \int\frac{d^3p}{(2\pi\hbar)^3}\int\frac{d^3q}{(2\pi\hbar)^3}\,e^{-i\frac{p\cdot x}{\hbar}}\,e^{i\frac{q\cdot y}{\hbar}}$$

$$\frac{1}{N!}\,N\,(N-1)!\,\delta^3(p)\,\delta^3(q)\,(2\pi\hbar)^6\,\underbrace{\langle gnd|gnd\rangle}_{1}$$

$$= N$$

$\Rightarrow$ It is as if $\langle gnd|\,a(x)\,|gnd\rangle = \sqrt{N}$ Condensate!