L 26

The Quantized Electromagnetic Field

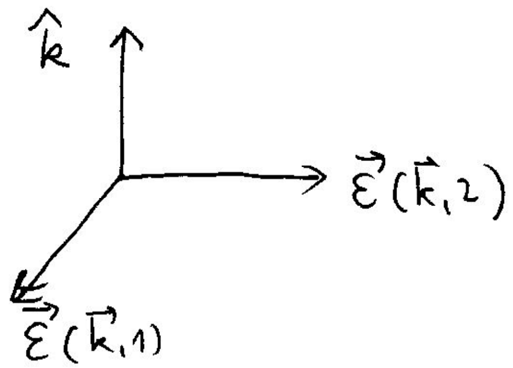
We have observed before that the energy of the electromagnetic field in the Coulomb gauge ($\vec{\nabla}\cdot\vec{A}=0$) is

$$H=\int d^3x \;\frac{1}{8\pi}\left[\frac{1}{c^2}\left(\frac{\partial\vec{A}}{\partial t}\right)^2+\left(\vec{\nabla}\times\vec{A}\right)^2\right]+\text{Coulomb Energy}.$$

In the term in brackets, the first term plays a role of a kinetic energy whereas the second term is like a potential energy. This expression, being quadratic in the dynamical variable $\vec{A}$ and its time derivative $\frac{\partial\vec{A}}{\partial t}$, resembles a problem of (many) harmonic oscillators. Indeed, ~~if~~ we can Fourier expand $\vec{A}(\vec{x},t)$

$$\vec{A}(\vec{x},t)=\int\frac{d^3k}{(2\pi)^3}\sum_{\lambda=1,2}\vec{\varepsilon}(\vec{k},\lambda)\,A(\vec{k},\lambda,t)\,e^{i\vec{k}\cdot\vec{x}}$$

where $\vec{\varepsilon}(\vec{k},\lambda)$ are the polarizations



$$\hat{k}=\vec{k}/|\vec{k}|$$

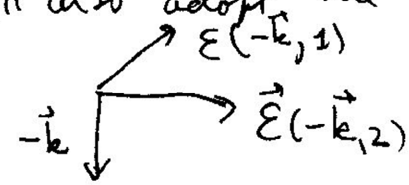
$$\vec{\varepsilon}(\vec{k},\lambda)\cdot\vec{\varepsilon}(\vec{k},\lambda')=\delta_{\lambda\lambda'}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A}=0 \quad\text{since}\quad \vec{k}\cdot\vec{\varepsilon}(\vec{k},\lambda)=0$$

We will also adopt the convention

$$\vec{\varepsilon}(-\vec{k},1)=-\vec{\varepsilon}(\vec{k},1)$$

$$\vec{\varepsilon}(-\vec{k},2)=+\vec{\varepsilon}(\vec{k},2)$$



$$\Rightarrow \varepsilon(-\vec{k},\lambda)=(-1)^{\lambda}\,\vec{\varepsilon}(\vec{k},\lambda)$$

$$\Rightarrow \quad \vec{\varepsilon}(\vec{k},\lambda)\cdot\vec{\varepsilon}(-\vec{k},\lambda') = (-1)^{\lambda}\,\delta_{\lambda\lambda'}$$

From the requirement that $\vec{A}(\vec{x},t)$ is real we then find that the amplitudes $A(\vec{k},\lambda,t)$ obey

$$A(\vec{k},\lambda,t)^{*} = (-1)^{\lambda}\, A(-\vec{k},\lambda,t)$$

We can now write the energy in terms of the amplitudes $A(\vec{k},\lambda,t)$. We find, after making extensive use of $\int d^3x \; e^{i\vec{k}\cdot\vec{x}} = (2\pi)^3\,\delta^3(\vec{k})$,

$$H = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda=1,2} \frac{1}{8\pi}\left(\frac{1}{c^2}\left| \frac{\partial A}{\partial t}(\vec{k},\lambda,t)\right|^2 + \vec{k}^2\,|A(\vec{k},\lambda,t)|^2 \right)$$

+ Coulomb terms.

In other words, for each mode $\vec{k}$ and polarization λ, we have a harmonic oscillator.

QM of Many Oscillators

This problem is very similar to the following one. Consider a system of N harmonic oscillators $\{Q_i\}$ $(i=1,\dots,N)$ with Hamiltonian

$$H = \sum_{i=1}^{N} \frac{1}{2m_i}\hat{P}_i^{\,2} + \sum_{i,j} V_{ij}\,\hat{Q}_i\hat{Q}_j$$

$$V_{ij} = V_{ji}$$

$$q_i = \sqrt{m_i}\,Q_i \qquad p_i = \frac{1}{\sqrt{m_i}}P_i, \quad U_{ij} = \frac{2}{\sqrt{m_i m_j}}V_{ij}$$

The oscillator coordinates and momenta obey the commutation relations (canonical)

$$[\hat{P}_i, \hat{P}_j] = [\hat{Q}_i, \hat{Q}_j] = 0$$

$$[\hat{Q}_i, \hat{P}_j] = i\hbar\, \delta_{ij}$$

$$\Rightarrow \quad [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0 \quad \text{and} \quad [\hat{q}_i, \hat{p}_j] = i\hbar\, \delta_{ij}$$

$$\Rightarrow H = \sum_i \frac{1}{2} \hat{p}_i^{\,2} + \sum_{ij} \frac{1}{2} U_{ij} \hat{q}_i \hat{q}_j$$

For a single oscillator

$$H_1 = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{q}^2$$

we defined

$$\hat{a} = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{m\omega}{\hbar}}\, \hat{q} + i \frac{1}{\sqrt{m\omega\hbar}}\, \hat{p} \right) \qquad \hat{q} = \sqrt{\frac{\hbar}{m\omega}}\, \frac{(\hat{a} + \hat{a}^+)}{\sqrt{2}}$$

$$\hat{a}^+ = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{m\omega}{\hbar}}\, \hat{q} - i \frac{1}{\sqrt{m\omega\hbar}}\, \hat{p} \right) \qquad \hat{p} = \sqrt{m\omega\hbar}\, \frac{(\hat{a} - \hat{a}^+)}{\sqrt{2}\, i}$$

$$[\hat{a}, \hat{a}^+] = 1$$

$$\Rightarrow H_1 = \frac{\hbar\omega}{2} \left(\hat{a}^+ \hat{a} + \hat{a} \hat{a}^+ \right) = \hbar\omega \left(\hat{a}^+ \hat{a} + \frac{1}{2} \right)$$

If $|0\rangle$, $\hat{a}|0\rangle = 0$ and $|n\rangle = \dfrac{(\hat{a}^+)^n}{\sqrt{n!}} |0\rangle$

$\Rightarrow H|n\rangle = \hbar\omega(n + \frac{1}{2})\, |n\rangle$ are the eigenstates.

For N oscillators we go to normal modes first and define

$$\hat{\tilde{q}}_\alpha = \sum_{i=1}^N C_{\alpha i} \hat{q}_i$$, where $\alpha = 1, \dots, N$ labels the N normal modes.

If U_{ij} is real symmetric $\Rightarrow$ $C_{\alpha i}$ must be orthogonal

$$\sum_{i=1}^N C_{\alpha i} C_{\beta i} = \delta_{\alpha\beta} \qquad \sum_{\alpha=1}^N C_{\alpha i} C_{\alpha j} = \delta_{ij}$$

$$\hat{q}_i = \sum_{\alpha=1}^N C_{i\alpha} \tilde{q}_\alpha$$

If U_{ij} is positive def. (i.e. it has positive e.v.'s)

$$\Rightarrow \sum_{i,j=1}^N U_{ij} C_{\alpha i} C_{\beta j} = \omega_\alpha^2 \delta_{\alpha\beta}$$

eigenvalues of U_{ij}

$$\Rightarrow \sum_{i,j=1}^N U_{ij} \hat{q}_i \hat{q}_j = \sum_\alpha \omega_\alpha^2 \hat{\tilde{q}}_\alpha^2$$

Define $\hat{\tilde{p}}_\alpha = \sum_{i=1} C_{\alpha i} \hat{p}_i$

$$\Rightarrow [\hat{\tilde{q}}_\alpha, \hat{\tilde{p}}_\beta] = \sum_{i,j=1}^N C_{\alpha i} C_{\beta j} [\hat{q}_i, \hat{p}_j] =$$

$$= i\hbar \sum_{i=1}^N C_{\alpha i} C_{\beta i} = i\hbar \delta_{\alpha\beta}$$ (canonical)

(canonical transformation)

$$\Rightarrow H = \sum_{\alpha=1}^N \frac{1}{2}\left(\hat{\tilde{p}}_\alpha^2 + \omega_\alpha^2 \hat{\tilde{q}}_\alpha^2 \right)$$

$$\hat{a}_\alpha = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{\omega_\alpha}\, \hat{\tilde{q}}_\alpha + \frac{i}{\sqrt{\omega_\alpha}} \hat{\tilde{p}}_\alpha \right)$$

$$\hat{a}_\alpha^\dagger = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{\omega_\alpha}\, \hat{\tilde{q}}_\alpha - \frac{i}{\sqrt{\omega_\alpha}} \hat{\tilde{p}}_\alpha \right)$$

$$\hat{\tilde{q}}_\alpha = \sqrt{\frac{\hbar}{2\omega_\alpha}} \left(\hat{a}_\alpha + \hat{a}_\alpha^\dagger \right)$$

$$\hat{\tilde{p}}_\alpha = -i \sqrt{\frac{\hbar\omega_\alpha}{2}} \left(\hat{a}_\alpha - \hat{a}_\alpha^\dagger \right)$$

with $\quad [\hat{a}_\alpha, \hat{a}_\beta] = [\hat{a}_\alpha^\dagger, \hat{a}_\beta^\dagger] = 0$

$$[\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta}$$

and $\quad H = \sum_\alpha \hbar\omega_\alpha \left(\hat{a}_\alpha^\dagger \hat{a}_\alpha + \frac{1}{2} \right)$

$\Rightarrow$ Let $|0\rangle = |0, \dots, 0\rangle$ be the state annihilated by

<u>all</u> the $\hat{a}_\alpha$'s

$$\hat{a}_\alpha |0\rangle = 0 \qquad \forall \alpha = 1, \dots, N$$

and

$$|n_1, \dots, n_N\rangle \equiv \left(\prod_{\alpha=1}^{N} \frac{(\hat{a}_\alpha^\dagger)^{n_\alpha}}{\sqrt{n_\alpha !}} \right) |0\rangle$$

$$\Rightarrow H |n_1, \dots, n_N\rangle = \sum_\alpha \hbar\omega_\alpha (n_\alpha + \tfrac{1}{2}) |n_1, \dots, n_N\rangle$$

$$E(n_1, \dots, n_N) = \sum_{\alpha=1}^{N} \hbar\omega_\alpha n_\alpha + \left(\frac{1}{2} \sum_{\alpha=1}^{N} \hbar\omega_\alpha \right)$$

$$E_0 = E(0, \dots, 0) = \sum_{\alpha=1}^{N} \tfrac{1}{2}\hbar\omega_\alpha \qquad \text{gnd. state}.$$

$$\Rightarrow E(n_1, \dots, n_N) = \sum_{\alpha=1}^{N} \hbar\omega_\alpha n_\alpha + E_0$$

Each oscillator has a level spacing $\hbar\omega_\alpha$ and excitation index (i.e. level) n_α.

Notice that for 1 oscillator

$$\hat{a}^{+}|n\rangle = \sqrt{n+1}\ |n+1\rangle$$

$$\hat{a}\ |n\rangle = \sqrt{n}\ \ |n-1\rangle$$

For N oscillators:

$$\Rightarrow \qquad \hat{a}^{+}_{\alpha}|n_1,..n_\alpha,..n_N\rangle = \sqrt{n_\alpha+1}\ |n_1,...,n_\alpha+1,...,n_N\rangle$$

$$\hat{a}_{\alpha}|n_1,..,n_\alpha,...n_N\rangle = \sqrt{n_\alpha}\ \ |n_1,...,n_\alpha-1,...n_N\rangle$$

where $\alpha=1,\dots,N$ label the ~~N~~ normal modes.

The expression for the energy of the state $|n_1,\dots,n_\alpha,\dots,n_N\rangle$,

$$E(n_1,\dots,n_\alpha,\dots,n_N) = \sum_{\alpha=1}^{N} \hbar\omega_\alpha n_\alpha + E_0$$

$$E_0 = \sum_{\alpha=1}^{m} \frac{1}{2}\hbar\omega_\alpha$$

shows that this state can be viewed as a collection of $\sum_{\alpha=1}^{N} n_\alpha$ objects where each ~~with~~ object is placed in a state $|\alpha\rangle$ with energy $\hbar\omega_\alpha$. The state with no objects has $n_\alpha=0$ (for all α) and its energy is E_0. We will call this state the <u>vacuum</u>, $|0\rangle = |0,\dots,0\rangle$. The objects are usually called <u>quanta</u> and there are N types of quanta, ~~each~~ labelled by $\alpha=1,\dots,N$.

A state with one quantum of type α

$|0,\dots,1_\alpha,\dots,0\rangle \equiv |\alpha\rangle$ has energy $\hbar\omega_\alpha + E_0$

Then the excitation energy is $E(0,\dots,1_\alpha,\dots,0) - E_0 = \hbar\omega_\alpha$

The labels n_α are then called <u>occupation numbers</u>.

One important property of ~~these~~ states

$$|n_1,\dots,n_\alpha,\dots,n_N\rangle = \prod_{\alpha=1}^{n} \frac{(\hat{a}_\alpha^+)^n}{\sqrt{n_\alpha!}}|0\rangle \quad \text{is}$$

their permutation symmetry $n_\alpha \leftrightarrow n_\beta$

This property follows from $[\hat{a}_\alpha^+, \hat{a}_\beta^+] = 0$

$\Rightarrow$ the wave functions are <u>symmetric</u> under pair-wise permutations of the labels (i.e. objects)

L22 Back to the Electromagnetic Field

We saw that the classical energy is

$$H = \int\frac{d^3k}{(2\pi)^3}\sum_{\lambda=1,2}\frac{1}{8\pi}\left(\frac{1}{c^2}\left|\frac{\partial A}{\partial t}(\vec{k},\lambda,t)\right|^2 + \vec{k}^2\,|A(\vec{k},\lambda,t)|\right)$$

where $A(-\vec{k},\lambda,t) = (-1)^\lambda A(\vec{k},\lambda,t)^*$ and

$$\vec{A}(\vec{x},t) = \sum_{\lambda=1,2}\int\frac{d^3k}{(2\pi)^3}\,\vec{\mathcal{E}}(\vec{k},\lambda)\,A(\vec{k},\lambda,t)\,e^{i\vec{k}\cdot\vec{x}}$$

$\lambda = 1,2$ labels the two polarization states.

Clearly we have two oscillators ($\lambda=1,2$) for each $\vec{k}$.

[Notice that both ~~the~~ real and imaginary parts of $A(\vec{k},\lambda,t)$ are not independent and that only one of them is sufficient to know the solution of the classical wave equation].

Also $\quad \Box \vec{A} = 0 \quad$ implies that

$$\Sigma_{\lambda=1,2} \int \frac{d^3k}{(2\pi)^3} \vec{\mathcal{E}}(\vec{k},\lambda) \left[-\vec{k}^2 A(\vec{k},\lambda,t) - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2}(\vec{k},\lambda,t) \right] e^{i\vec{k}\cdot\vec{x}} = 0$$

$$\Rightarrow \quad -\vec{k}^2 A(\vec{k},\lambda,t) = \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2}(\vec{k},\lambda,t)$$

$$\Rightarrow \quad \vec{A}(\vec{k},\lambda,t) = \vec{A}(\vec{k},\lambda)\, e^{-i\omega t} \qquad \text{with} \quad \omega = |\vec{k}|c$$

and ω can be regarded as the frequency of the oscillator of type $(\vec{k},\lambda)$.

We quantize ~~the~~ the e.m. field by introducing a set of creation and annihilation operators (or raising and lowering ops.) $\quad \hat{a}^+(\vec{k},\lambda)$, $\hat{a}(\vec{k},\lambda)$ and write (with a slightly different normalization)

$$\hat{\vec{A}}(\vec{x},t) = \int \frac{d^3k \sqrt{4\pi}\, \hbar c}{\sqrt{(2\pi)^3\, 2E(\vec{k})}} \sum_{\lambda=1,2} \vec{\mathcal{E}}(\vec{k},\lambda) \left[\hat{a}(\vec{k},\lambda)\, e^{-ik\cdot x} + \hat{a}^+(\vec{k},\lambda)\, e^{ik\cdot x} \right]$$

where $\quad E(\vec{k}) = \hbar\omega(\vec{k}) = \hbar c|\vec{k}|$

$$k\cdot x = \omega t - \vec{k}\cdot\vec{x}$$

[this normalization is motivated by the need to satisfy relativistic covariance] where the operators $\hat{a}(\vec{k},\lambda)$, $\hat{a}^{+}(\vec{k},\lambda)$ obey the commutation relations

$$[\hat{a}(\vec{k},\lambda), \hat{a}(\vec{k}',\lambda')] = [\hat{a}^{+}(\vec{k},\lambda), \hat{a}^{+}(\vec{k}',\lambda')] = 0$$

$$[\hat{a}(\vec{k},\lambda), \hat{a}^{+}(\vec{k}',\lambda')] = \delta^3(\vec{k}-\vec{k}')\,\delta_{\lambda\lambda'}$$

In terms of these operators, the Hamiltonian is

$$H = \int d^3x \, \frac{1}{8\pi}\left(\vec{E}^2 + \vec{B}^2\right) \equiv$$

$$\equiv \int d^3k \; \frac{1}{2} E(\vec{k}) \sum_{\lambda} \left[\hat{a}(\vec{k},\lambda)\,\hat{a}^{+}(\vec{k},\lambda) + \hat{a}^{+}(\vec{k},\lambda)\,\hat{a}(\vec{k},\lambda)\right]$$

$$\equiv \int d^3k \sum_{\lambda=1,2} E(\vec{k})\,\hat{a}^{+}(\vec{k},\lambda)\,\hat{a}(\vec{k},\lambda) +$$

$$+ \int d^3k \sum_{\lambda=1,2} E(\vec{k})\,\delta_{\lambda,\lambda}\,\delta^3(\vec{k}-\vec{k})$$

But $\delta^3(k=0) = \left(\frac{L}{2\pi}\right)^3 = \frac{\text{Volume}}{(2\pi)^3} = \frac{V}{(2\pi)^3}$

$$H = \int d^3k \sum_{\lambda=1,2} E(\vec{k})\,\hat{a}^{+}(\vec{k},\lambda)\,\hat{a}(\vec{k},\lambda) + E_0$$

$$E_0 = \frac{1}{2} V \int \frac{d^3k}{(2\pi)^3} E(\vec{k}) \quad (= \infty!)$$ = sum of the gnd state energies of all the oscillators.

$$H = \int d^3k \sum_{\lambda} E(\vec{k})\,\hat{N}(\vec{k},\lambda) + E_0$$

Photon # operator $\hat{N}(\vec{k},\lambda) \equiv \hat{a}^{+}(\vec{k},\lambda)\,\hat{a}(\vec{k},\lambda)$

Vacuum $|0\rangle = |0, \dots, 0_{\vec{k},\lambda}, \dots 0\rangle$

$\Rightarrow \quad \hat{a}(\vec{k},\lambda)\,|0\rangle = 0$ No photons

We find a quanta (or photon) for each $\vec{k}$ and λ

$$|\vec{k},\lambda\rangle = \hat{a}^\dagger(\vec{k},\lambda)\,|0\rangle = |0, \dots, 1_{\vec{k},\lambda}, \dots, 0\rangle$$

$$H\,|k,\lambda\rangle = E(\vec{k})\;|\vec{k},\lambda\rangle$$

$\Rightarrow$ this state has one photon of momentum $\hbar\vec{k}$, ~~and~~ polarization λ and energy $\vec{E}(k,\lambda) = E(\vec{k}) = \hbar c|\vec{k}|$

$\Rightarrow$ for each momentum $\hbar\vec{k}$ there are two photon states with the same energy but $\neq$ polarization.

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Total momentum:

$$\hat{\vec{P}} = \int d^3x\,\frac{1}{c^2}\hat{\vec{\mathcal{P}}}(\vec{x}) = \int d^3x\;\frac{1}{4\pi c}\left(\vec{E}(\vec{x},t)\times\vec{B}(\vec{x},t)\right)$$

$$\equiv -\int d^3x\;\frac{1}{4\pi c}\;\frac{\partial\vec{A}}{\partial t}(\vec{x},t)\times\left(\vec{\nabla}\times\vec{A}(\vec{x},t)\right)$$

$$= \int d^3k\;\hbar\vec{k}\;\sum_\lambda \hat{a}^\dagger(\vec{k},\lambda)\;\hat{a}(\vec{k},\lambda)$$

$\Rightarrow \quad \hat{\vec{P}}\,|0\rangle = 0$ and

$$\hat{\vec{P}}\,|\vec{k},\lambda\rangle = \hat{\vec{P}}\,\hat{a}^\dagger(\vec{k},\lambda)\,|0\rangle = \hbar\vec{k}\;|\vec{k},\lambda\rangle \qquad \text{(for both } \lambda=1,2)$$

Similarly the angular momentum of the e.m. field is

$$\vec{L} = \int d^3x \, \frac{1}{4\pi c} \, \vec{r} \times (\vec{E} \times \vec{B}) = -\int d^3x \, \frac{1}{4\pi c^2} \, \vec{r} \times \left(\frac{\partial \vec{A}}{\partial t} \times (\vec{\nabla} \times \vec{A}) \right)$$

$$L_a = -\frac{1}{4\pi c^2} \int d^3x \left[\frac{\partial \overleftrightarrow{A}}{\partial t} \cdot \left(\vec{r} \times \frac{\partial}{\partial \vec{r}} \right)_a \vec{A} + \left(\frac{\partial \vec{A}}{\partial t} \times \vec{A} \right)_a \right]$$

The first term in brackets is similar to the usual orbital angular momentum and it vanishes when acting on plane wave states. The last term is directly related to the polarization of the state and it tells us that there is a relation between the polarization and angular momentum. Let's examine the meaning of this last term, which we will call $\hat{M}_a$,

$$\hat{M}_a = -\frac{1}{4\pi c^2} \int d^3x \left(\frac{\partial \vec{A}}{\partial t} \times \vec{A} \right)_a =$$

$$= \hbar \int d^3k \, \frac{k_a}{|\vec{k}|} \, (-i) \left[\hat{a}^+(k,1) \, \hat{a}(k,2) - \hat{a}^+(k,2) \, \hat{a}(k,1) \right]$$

Define

$$\hat{a}^+_R(k) = \frac{1}{\sqrt{2}} \left[\hat{a}^+(k,1) + i \, \hat{a}^+(k,2) \right] \qquad \hat{a}^+(k,1) = \frac{a^+_R + a^+_L}{\sqrt{2}}$$

$$\hat{a}^+_L(k) = \frac{1}{\sqrt{2}} \left[\hat{a}^+(k,1) - i \, \hat{a}^+(k,2) \right] \qquad \hat{a}^+(k,2) = \frac{a_R - a^+_L}{i\sqrt{2}}$$

[Check: $\hat{a}^+_R(k) \, \hat{a}_R(k) + \hat{a}^+_L(k) \, \hat{a}_L(k) = \sum_{\lambda=1,2} \hat{a}^+(k,\lambda) \, \hat{a}(k,\lambda)$]

$-i \, (a^+(k,1) \, a(k,2) - a^+(k,2) \, a(k,1) = a^+_R(k) \, a_R(k) - a^+_L(k) \, a_L(k)$

$$\hat{M}_a = \int d^3k\ \hbar \hat{k}_a \left[\hat{a}^\dagger_R(\vec{k})\, \hat{a}_R(\vec{k}) - \hat{a}^\dagger_L(\vec{k})\, \hat{a}_L(\vec{k}) \right]$$
$$\equiv \int d^3k\ \hbar \hat{k}_a \left[\hat{N}(\vec{k},R) - \hat{N}(\vec{k},L) \right]$$

The states $|\vec{p},R\rangle$ and $|\vec{p},L\rangle$ are eigenstates of $\hat{M}_a$

$$\hat{M}_a |\vec{p},R\rangle = \hat{M}_a\, \hat{a}^\dagger(\vec{p},R)|0\rangle = \hat{M}_a\ \hat{a}_R(\vec{p})|0\rangle$$
$$= \int d^3k\ \hbar\hat{k}_a \left[\hat{a}^\dagger_R(\vec{k})\ \hat{a}_R(\vec{k}) - \hat{a}^\dagger_L(\vec{k})\, \hat{a}_L(\vec{k}) \right] \hat{a}^\dagger(\vec{p},R)|0\rangle$$

But $\hat{M}_a|0\rangle = \int d^3k\ \hbar\hat{k}_a \left[\hat{a}^\dagger_R(\vec{k})\ \hat{a}_R(\vec{k}) - \hat{a}^\dagger_L(\vec{k})\ \hat{a}_L(\vec{k}) \right]|0\rangle = 0$

(i.e. $|0\rangle$ has zero angular momentum)

and $\left[\hat{M}_a, \hat{a}^\dagger_R(\vec{p}) \right] = \int d^3k\ \hbar\, \hat{k}_a \Big\{ \left[\hat{a}^\dagger_R(\vec{k})\, \hat{a}_R(\vec{k}), \hat{a}^\dagger_R(\vec{p}) \right] -$
$$- \left[\hat{a}^\dagger_L(\vec{k})\, \hat{a}_L(\vec{k}), \hat{a}^\dagger_R(\vec{k}) \right] \Big\}$$

Since $\left[\hat{a}^\dagger_L(\vec{k}), \hat{a}^\dagger_R(\vec{p}) \right] = \left[\hat{a}_L(\vec{k}), \hat{a}^\dagger_R(\vec{p}) \right] = 0$

Also, ~~and~~ using that $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$, we get

$$\left[\hat{a}^\dagger_R(\vec{k})\ \hat{a}_R(\vec{k}), \hat{a}^\dagger_R(\vec{p}) \right] = \hat{a}^\dagger_R(\vec{k}) \left[\hat{a}_R(\vec{k}), \hat{a}^\dagger_R(\vec{p}) \right]$$
$$+ \left[\hat{a}^\dagger_R(\vec{k}), \hat{a}^\dagger_R(\vec{p}) \right] \hat{a}_R(\vec{k})$$
$$= \delta^{(3)}(\vec{k}-\vec{p})\ \hat{a}^\dagger_R(\vec{k})$$

$$\left[\hat{a}^\dagger_L(\vec{k})\ \hat{a}_L(\vec{k}), \hat{a}^\dagger_R(\vec{p}) \right] = 0$$

$$\Rightarrow \left[\hat{M}_a, \hat{a}^\dagger_R(\vec{p}) \right] = \int d^3k\ \hbar\hat{k}_a\ \delta^3(\vec{k}-\vec{p}) = \hbar\hat{p}_a \equiv \hbar \frac{\vec{p}_a}{|\vec{p}|}$$

Similarly, we also get

$$\left[\hat{M}_a, \hat{a}^\dagger_L(\vec{p})\right] = -\hbar\, \hat{p}_a$$

Then

$$\hat{M}_a |\vec{p}, R\rangle = \hbar\, \hat{p}_a |\vec{p}, R\rangle \qquad\qquad \hat{p} = \vec{p}/|\vec{p}|$$

$$\hat{M}_a |\vec{p}, L\rangle = -\hbar\, \hat{p}_a |\vec{p}, L\rangle$$

We conclude that the photon states carry an intrinsic angular momentum $\pm\hbar$, parallel or antiparallel to the direction of the momentum $\vec{p}$. When the angular momentum is either parallel or antiparallel to the momentum, it is usually called the helicity of the state

Q: Since the $\vec{p}$ projection of $\hat{M}$ is $\pm\hbar$ we should expect to find a state with zero angular momentum. Why isn't it there?

A: The e.m. field in vacuum is purely transverse and it has only two physical components, each with $\neq 0$ polarization $\Rightarrow$ $\Rightarrow$ $\neq 0$ angular momentum.

In this basis the mode expansion

$$\vec{A}(\vec{x},t) = \int \frac{d^3k \sqrt{4\pi}\,\hbar c}{\sqrt{(2\pi)^3\, 2E(\vec{k})}} \sum_{\lambda=1,2} \Big[\vec{\varepsilon}(k,\lambda)\, \hat{a}(\vec{k},\lambda)\, e^{+i\vec{k}\cdot\vec{x}} + $$

$$+ \vec{\varepsilon}(\vec{k},\lambda)\, \hat{a}^\dagger(\vec{k},\lambda)\, e^{-i\vec{k}\cdot\vec{x}} \Big]$$

can be written in terms of the R and L polarization vectors

$$\vec{\varepsilon}(\vec{k},R) \equiv \frac{1}{\sqrt{2}} \left(\vec{\varepsilon}(\vec{k},1) - i\, \vec{\varepsilon}(\vec{k},2) \right)$$

$$\vec{\varepsilon}(\vec{k},L) \equiv \frac{1}{\sqrt{2}} \left(\vec{\varepsilon}(k,1) + i\, \varepsilon(\vec{k},2) \right)$$

$$\Rightarrow \vec{A}(\vec{x},t) = \int \frac{d^3k \sqrt{4\pi}\,\hbar c}{\sqrt{(2\pi)^3\, 2E(\vec{k})}} \sum_{\lambda=R,L} \vec{\varepsilon}(\vec{k},\lambda) \Big[\hat{a}(\vec{k},\lambda)\, e^{+i\vec{k}\cdot\vec{x}} + \hat{a}^\dagger(\vec{k},\lambda)\, e^{-i\vec{k}\cdot\vec{x}} \Big]$$

Notice also that

$$\vec{\varepsilon}(-\vec{k},R) = -\vec{\varepsilon}(\vec{k},L)$$

$$\vec{\varepsilon}(-\vec{k},L) = -\varepsilon(\vec{k},R)$$

$$\vec{\varepsilon}^{\,*}(\vec{k},R) = \vec{\varepsilon}(\vec{k},L)$$ and viceversa.