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Interaction of Radiation with matter

We will discuss the QM description of the interaction of matter with radiation. A full description requires ~~the~~ a knowledge of QED which we won't do here. Instead, we will describe

(a) the classical picture (review).

(b) quantized matter (not field theory) interacting with classical E.M. radiation (at the level of leading orders in perturbation theory.

(c) non-relativistic quantized matter interacting with the quantized e.m. field at leading orders in p.th.

Review of Classical E.M.

The electromagnetic field, as a classical dynamical problem, is described by the electric and magnetic fields $\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$ which are the solutions of Maxwell's equations which, in gaussian units, are

$$\vec{\nabla}\cdot\vec{E}(\vec{r},t) = 4\pi\,\rho(\vec{r},t) \qquad \vec{\nabla}\cdot\vec{B}(\vec{r},t) = 0$$

$$\vec{\nabla}\times\vec{B}(\vec{r},t) = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{j}(\vec{r},t) \qquad \vec{\nabla}\times\vec{E}(\vec{r},t) = -\frac{1}{c}\frac{\partial\vec{B}(\vec{r},t)}{\partial t}$$

where $\rho(\vec{r},t)$ and $\vec{j}(\vec{r},t)$ are the charge density and current density and ~~satisfy~~ satisfy the <u>continuity equation</u>

$$\frac{1}{c}\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

which expresses the local conservation of charge. The two M.E.'s on the r.h.s. are <u>constraints</u>. The first, $\vec{\nabla}\cdot\vec{B} = 0$ means that $\vec{B}$ has no sources or sinks (i.e. <u>no</u> <u>monopoles</u>) and can be satisfied if

$$\vec{B} = \nabla\times\vec{A}$$

where $\vec{A}$ is the <u>vector potential</u>.

The 2nd eqn. $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ then implies that

$$\vec{\nabla}\times\left[\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right] = 0$$

$$\Rightarrow \quad \vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi \qquad \phi:\ \underline{\text{scalar potential}}$$

$$\Rightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A} \quad , \quad \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi$$

However, there are <u>many</u> $(\phi, \vec{A})$ that describe the same electric and magnetic fields, namely

$$\left.\begin{aligned} \vec{A} &\to \vec{A}' = \vec{A} + \vec{\nabla}\chi \\ \phi &\to \phi' = \phi - \frac{1}{c}\frac{\partial \cancel{A}\chi}{\partial t} \end{aligned}\right] \qquad \begin{aligned} \vec{E} &\to \vec{E}' = \vec{E} \\ \vec{B} &\to \vec{B}' = \vec{B} \end{aligned}$$

gauge transformations.

[The scalar potential ϕ is usually denoted by A_0 and the four vector $(\frac{1}{c}A_0, \vec{A})$ transforms like a four vector under Lorentz transformations]

Classically it is often said that $\vec{E}$ and $\vec{B}$ are physical while $\vec{A}, \phi$ are not (are "auxiliary"). However in QM the vector potential becomes a natural physical quantity [Bohm-Aharonov effect].

The charge density and current density are due to ~~the particle~~ matter (i.e. particles). Thus, for N particles at locations $\vec{r}_i(t)$ $(i=1,\dots,N)$

$$\rho(\vec{r},t) = \sum_{i=1}^{N} q_i \, \delta(\vec{r} - \vec{r}_i(t))$$

where q_i are the charges of the particles.

The energy density carried by the field is

$$\mathcal{E}(\vec{r},t) = \frac{1}{8\pi}\left(\vec{E}^2(\vec{r},t) + \vec{B}^2(\vec{r},t)\right)$$

while the total energy is

$$E = \int_V d^3r \; \mathcal{E}(\vec{r},t) = \int_V d^3r \; \frac{1}{8\pi}\left[\vec{E}^2(\vec{r},t) + \vec{B}^2(\vec{r},t)\right]$$

The rate and direction of energy transport is given by the Poynting vector

$$\vec{\mathcal{P}}(\vec{r},t) = \frac{c}{4\pi}\left(\vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t)\right)$$

$$[\mathcal{P}] = (\text{ergs}/\text{cm}^2)/\text{sec}$$

We can now write Maxwell's eqns. in terms of $\vec{A}$ and ϕ.

$$\vec{\nabla}\cdot\left[-\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi\right] = 4\pi\rho$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \frac{1}{c}\frac{\partial}{\partial t}\left[-\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi\right] + \frac{4\pi}{c}\vec{J}$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

Using the gauge freedom we can impose the Lorentz condition (Lorentz gauge) (for example)

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$$

Then one finds that $\vec{A}$ and ϕ satisfy

$$\Box \vec{A} = -\frac{4\pi}{c}\vec{J}$$

$$\Box \phi = -4\pi\rho$$

(recall that ϕ and $\vec{A}$ are no longer independent!)

$$\Box \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \text{"D'Alambertian"}$$

In the abscence of matter $(\rho = 0)$ we ~~can~~ get

$$\phi = 0 \Rightarrow \frac{1}{c}\frac{\partial \phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$$

$$\phi = 0 \Rightarrow \vec{\nabla}\cdot\vec{A} = 0 \quad \text{(transversality)}$$

This is the so-called Coulomb gauge. $(\phi = \vec{\nabla}\cdot\vec{A} = 0)$

Another "popular" gauge is the transverse gauge

$$\vec{\nabla}\cdot\vec{A} = 0$$

~~In~~ this gauge, Gauss' Law

$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ becomes $\vec{\nabla}\cdot\left[-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right] = 4\pi\rho$

$$\Rightarrow -\nabla^2\phi = 4\pi\rho \quad \text{(purely static!)}$$

Let $G_0(\vec{r}-\vec{r}\,')$ be the 3D Green's function

$$-\nabla^2 G_0 = \delta(r-r') \qquad G_0(\vec{r}-\vec{r}\,') = \frac{1}{4\pi|\vec{r}-\vec{r}\,'|}$$

$$\phi(\vec{r}) = \int d^3r' \; G_0(\vec{r}-\vec{r}\,') \; 4\pi\rho(\vec{r}\,')$$

In this gauge the equation of motion becomes

$$\Box \vec{A} = \frac{1}{c} \vec{\nabla} \frac{\partial \phi}{\partial t} - \frac{4\pi}{c} \vec{J}$$

Since $\vec{\nabla} \cdot \vec{A} = 0$ we split $\vec{J} = \vec{J}_L + \vec{J}_T$

$$\vec{\nabla} \cdot \vec{J}_T = 0 \qquad \vec{\nabla} \times \vec{J}_L = 0$$

$\Rightarrow$ taking divergence

$$0 = \Box \vec{\nabla} \cdot \vec{A} = \frac{1}{c} \nabla^2 \frac{\partial \phi}{\partial t} - \frac{4\pi}{c} \vec{\nabla} \cdot \vec{J}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad \text{and} \quad -\nabla^2 \phi = 4\pi \rho$$

we find that the r.h.s is indeed zero.

$$\Rightarrow \boxed{\Box \vec{A} = -\frac{4\pi}{c} \vec{J}_T}$$

(in the transverse gauge) $\boxed{\vec{\nabla} \cdot \vec{A} = 0}$

What about the energy?

Total energy: $$E = \frac{1}{8\pi} \int d^3r \left(\vec{E}^2 + \vec{B}^2 \right)$$

$$= \frac{1}{8\pi} \int d^3r \left[\left(\frac{1}{c} \frac{\partial \vec{A}}{\partial t} + \vec{\nabla} \phi \right)^2 + \left(\vec{\nabla} \times \vec{A} \right)^2 \right]$$

$$= \frac{1}{8\pi} \int d^3r \left[\frac{1}{c^2} \left(\frac{\partial \vec{A}}{\partial t} \right)^2 + \left(\vec{\nabla} \times \vec{A} \right)^2 \right] +$$

$$+ \frac{1}{8\pi} \int d^3r \left(\vec{\nabla} \phi \right)^2 + \frac{1}{8\pi} \int d^3r \frac{2}{c} \frac{\partial \vec{A}}{\partial t} \cdot \vec{\nabla} \phi$$

$$\frac{1}{8\pi}\int d^3r\, (\vec\nabla\phi)^2 = -\frac{1}{8\pi}\int d^3r\, \phi\, \nabla^2\phi = +\frac{4\pi}{8\pi}\int d^3r\, \rho\,\phi$$

no charges at ∞

$$= \frac{1}{2}\int d^3r\, \rho(r,t)\phi(r,t)$$

But $\phi(r,t) = \int d^3r' \dfrac{\rho(\vec r',t)}{|\vec r-\vec r'|}$

$$\Rightarrow \frac{1}{8\pi}\int d^3r\,(\vec\nabla\phi)^2 = \frac{1}{2}\int d^3r\int d^3r' \frac{\rho(\vec r,t)\,\rho(\vec r',t)}{|\vec r-\vec r'|}$$

: instantaneous Coulomb interaction

$$\int d^3r\, \frac{\partial\vec A}{\partial t}\cdot\vec\nabla\phi = -\int d^3r\, \frac{\partial}{\partial t}(\vec\nabla\cdot\vec A)\,\phi = 0$$

$\Rightarrow$ total energy:

$$\boxed{E = \int d^3r\, \frac{1}{8\pi}\left[\frac{1}{c^2}\left(\frac{\partial\vec A}{\partial t}\right)^2 + (\vec\nabla\times\vec A)^2\right] + \frac{1}{2}\int d^3r\int d^3r' \frac{\rho(\vec r,t)\,\rho(\vec r',t)}{|\vec r-\vec r'|}}$$

where $\vec A$ is transverse, $\vec\nabla\cdot\vec A = 0$

Thus, in the transverse gauge, one solves the (transverse) wave equation

$$\Box\vec A = -\frac{4\pi}{c}\vec j_T \qquad \vec\nabla\cdot\vec A = 0$$

and the energy is given by the expression boxed above.

Skip to Quantization of Free Maxwell

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Consider a plane wave solution in the region of space where $\vec{J} = 0 \Rightarrow$

$$\vec{A}(\vec{r},t) = \alpha \vec{\lambda} e^{i(\vec{k}\cdot\vec{r} - \omega t)} + \alpha^* \vec{\lambda}^* e^{-i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\alpha \in \mathbb{C} \qquad \vec{\lambda} \in \mathbb{C}^3$$

$$\Rightarrow \quad \Box \vec{A} = \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = 0 \Rightarrow \quad \omega = c|\vec{k}|$$

and $\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \vec{\lambda}\cdot\vec{k} = 0$

$\vec{\lambda}$: polarization vector

$$\text{Energy density} = \frac{1}{8\pi}\left(\vec{E}^2 + \vec{B}^2\right) = \frac{\omega^2}{2\pi c^2}\left\{|\alpha|^2 - \mathrm{Re}\left(\alpha^2\lambda^2 e^{i(\vec{k}\cdot\vec{r} - i2\omega t)}\right)\right\}$$

oscillates and it averages to zero in one period.

$$\Rightarrow \quad \frac{\langle E\rangle_{\text{period}}}{\text{Vol}} = \frac{\omega^2}{2\pi c^2}|\alpha|^2$$

The wave travels at speed c in the direction of $\vec{k} \Rightarrow$
~~energy flux~~ time average of the Poynting vector is

$$\langle \vec{S} \rangle = \hat{k}\,\frac{\omega^2}{2\pi c}|\alpha|^2$$

A general e.m. wave in free space is a linear superposition

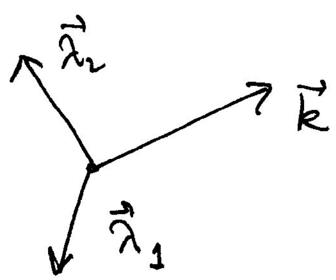
$$\cancel{\vec{A}(\vec{r},t) = \sum_{k,\vec{\lambda}} \left[A_{\vec{\lambda}}(\vec{k}) \right.}$$

$$\vec{A}(\vec{r},t) = \sum_{\vec{k},\vec{\lambda}} \left\{ \frac{A(\vec{k},\vec{\lambda})}{\sqrt{V}} \, \vec{\lambda} \, e^{i\vec{k}\cdot\vec{r} - i\omega t} + \right.$$

$$\left. + \frac{A(\vec{k},\vec{\lambda})^*}{\sqrt{V}} \, \vec{\lambda}^* \, e^{-i\vec{k}\cdot\vec{r} + i\omega t} \right\}$$

Once again $\quad \Box \vec{A} = 0 \Rightarrow \quad \omega = c|\vec{k}|$

$\vec{\lambda}$: two orthogonal polarizations for <u>each</u> $\vec{k}$.



$$\vec{\lambda}_i \cdot \vec{k} = 0$$

$$\vec{\lambda}_i \cdot \vec{\lambda}_j = \delta_{ij}$$

[for simplicity we can use PBC's]

The total energy is:

$$E = \int d^3r \, \frac{1}{8\pi} \left(\vec{E}^2 + \vec{B}^2 \right) = \sum_{\vec{k},\vec{\lambda}} \frac{\omega^2}{2\pi c^2} \left| A(\vec{k},\vec{\lambda}) \right|^2$$

<u>How about matter?</u>

The Lagrangian for <u>particles</u> $(\vec{r}_1, \dots, \vec{r}_N)$ interacting with an em field is

$$L = \sum_{i=1}^{N} \frac{1}{2} m \left(\frac{d\vec{r}_i}{dt} \right)^2 + \sum_{i=1}^{N} \frac{q_i}{c} \vec{A}(\vec{r}_i, t) \cdot \frac{d\vec{r}_i}{dt}$$

$$- \sum_{i=1}^{N} \left[V(\vec{r}_i) + q_i \, \phi(\vec{r}_i, t) \right]$$

$$\vec{p}_i = \frac{\partial L}{\partial \frac{d\vec{r}_i}{dt}} = m\frac{d\vec{r}_i}{dt} + \frac{q_i}{c}\vec{A}(\vec{r}_i, t)$$

$$\Rightarrow \quad \frac{d\vec{r}_i}{dt} = \frac{1}{m}\left[\vec{p}_i - \frac{q_i}{c}\vec{A}(\vec{r}_i, t)\right]$$

<u>Classical Hamiltonian:</u>

$$H = \sum_{i=1}^{N} \frac{1}{2m}\left(\vec{p}_i - \frac{q_i}{c}\vec{A}(\vec{r}_i, t)\right)^2 + \sum_{i=1}^{N}\left[V(\vec{r}_i) + q_i\,\phi(\vec{r}_i, t)\right]$$

<u>Quantization:</u> Schrödinger Eqn. (one particle, $q = e$)

$$i\hbar\frac{\partial \psi(\vec{r}, t)}{\partial t} = \left[\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 + e\,\phi(\vec{r}, t) + V(\vec{r})\right]\psi(\vec{r}, t)$$

Gauge invariance: $\vec{A} \to \vec{A} + \vec{\nabla}\chi = \vec{A}'$

$$\phi \to \phi - \frac{1}{c}\frac{\partial \chi}{\partial t} = \phi'$$

$\Rightarrow$ the wave function transforms as

$$\psi(\vec{r}, t) \to \psi'(\vec{r}, t) = e^{i\frac{e}{\hbar c}\chi(\vec{r}, t)}\,\psi(\vec{r}, t)$$

(pure phase)

The momentum op. $\vec{p}$ is <u>not</u> gauge invariant

$$\langle \psi_1 | \vec{p} | \psi_2 \rangle = \int d^3r\, \psi_1^*(\vec{r})\,\frac{\hbar}{i}\vec{\nabla}\,\psi_2(\vec{r})$$

$$\to \int d^3r\, \psi_1^*(r)\, e^{-\frac{ie}{\hbar c}\chi}\,\frac{\hbar}{i}\vec{\nabla}\left(e^{\frac{ie}{\hbar c}\chi}\psi_2\right)$$

$$= \int d^3r\, \psi_1^*\,\frac{\hbar}{i}\vec{\nabla}\,\psi_2 + \int d^3r\, \frac{e}{c}(\vec{\nabla}\chi)\,\psi_1^*\psi_2$$

but

$$\langle \psi_1 | \vec{p} - \frac{e}{c} \vec{A} | \psi_2 \rangle \text{ is gauge invariant.}$$

Thus, in the Heisenberg rep., one finds

$$m \frac{d\vec{r}}{dt} = \vec{p}(t) - \frac{e}{c} \vec{A}(\vec{r}(t), t)$$

and, even though the wave functions transform, the velocity does not. matrix elements of

Similarly, the action that enters in the path integral is

$$S = \int_i^f dt \left[\frac{1}{2} m \left(\frac{d\vec{r}}{dt} \right)^2 + \frac{e}{c} \vec{A} \cdot \frac{d\vec{r}}{dt} \right] + \cancel{\ldots}$$

$$+ \int_i^f dt \left[- V(\vec{r}) - e\, \phi(\vec{r}, t) \right]$$

under a gauge transformation $\vec{A} \to \vec{A} + \vec{\nabla} \chi$

$$\phi \to \phi - \frac{1}{c} \frac{\partial \chi}{\partial t}$$

$\Rightarrow$ S changes by

$$S \to S + \int_i^f dt \left[\frac{e}{c} \vec{\nabla} \chi \cdot \frac{d\vec{r}}{dt} - e \left(- \frac{1}{c} \frac{\partial \chi}{\partial t} \right) \right]$$

$$= S + \int_i^f \frac{e}{c} \left(\vec{\nabla} \chi \cdot d\vec{r} + \frac{\partial \chi}{\partial t} dt \right)$$

$$d\chi = \vec{\nabla} \chi \cdot d\vec{r} + \frac{\partial \chi}{\partial t} dt$$

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$$\Rightarrow \quad S \to S + \frac{e}{c}\int_i^f d\chi \;\Rightarrow\; \Delta S = \frac{e}{c}\left[\chi(f) - \chi(i)\right]$$

$\Rightarrow$ the weight of the path integral changes by

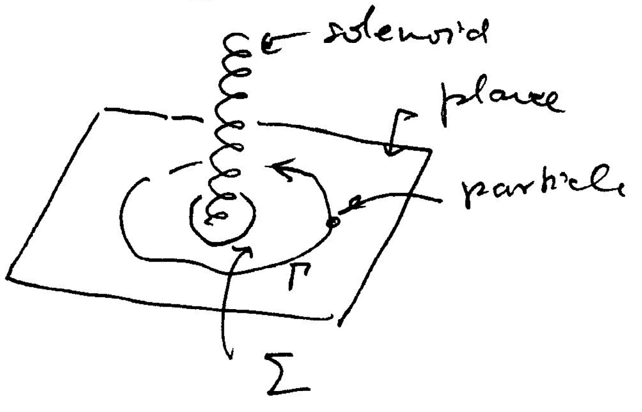
$$e^{\frac{i}{\hbar}S} \to e^{\frac{i}{\hbar}S}\, e^{\frac{i}{\hbar}\frac{e}{c}\left[\chi(f)-\chi(i)\right]}$$

Suppose we look at amplitudes $|\vec{R}, t_i\rangle \to |\vec{R}, t_f\rangle$

(i.e. $\vec{r}_i = \vec{r}_f = \vec{R}$) $\Rightarrow$ this factor does not change the amplitude if $\chi_f - \chi_i = 2\pi \frac{\hbar c}{e} n$ $\quad (n \in \mathbb{Z})$

However we will see now that, under proper circumstances, these effects are observable.

Example:



where the particle is there is no field, $\vec{B} = 0$

but $\vec{A} \neq 0$

Since $\oint_\Gamma \vec{A}\cdot d\vec{r} = \iint_\Sigma d\vec{S}\cdot\vec{B}$

$(\vec{B} = \vec{\nabla}\times\vec{A})$

$$\oint_\Gamma \vec{A}\cdot d\vec{r} = \Phi \quad \text{flux inside the solenoid.}$$

$\vec{A} = (A_\rho, A_\varphi, A_z)$ cylindrical coords.

$$\oint_C \vec{A}\cdot d\vec{r} = \int_0^{2\pi} r\,d\varphi\, A_\varphi = \Phi \;\Rightarrow\; A_\varphi = \frac{\Phi}{2\pi r}$$

$A_\rho = A_z = 0$

$$A_x = -\frac{\Phi}{2\pi}\frac{y}{x^2+y^2} = \frac{\Phi}{2\pi}\frac{\partial}{\partial x}\tan^{-1}\left(\frac{y}{x}\right)$$

$$A_y = +\frac{\Phi}{2\pi}\frac{x}{x^2+y^2} = \frac{\Phi}{2\pi}\frac{\partial}{\partial y}\tan^{-1}\left(\frac{y}{x}\right)$$

$$\Rightarrow \vec{A} = (A_x, A_y, A_z) = \frac{\Phi}{2\pi}\vec{\nabla}\tan^{-1}\left(\frac{y}{x}\right)$$

$\Rightarrow \vec{\nabla}\times\vec{A} = 0$ except at $(x,y)=0$

$\Rightarrow \vec{A} \equiv 0$ up to a gauge transformation

$$\chi = \frac{\Phi}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) \equiv \frac{\Phi}{2\pi}\arg(\vec{r})$$

but

$$\Delta\chi = \frac{\Phi}{2\pi}\times 2\pi = \Phi$$

$$\Rightarrow e^{i\frac{S}{\hbar}} \to e^{i\frac{S}{\hbar}}\, e^{\frac{i}{\hbar}\frac{e}{c}\Delta\chi} =$$

$$= e^{i\frac{S}{\hbar}}\, e^{\frac{i}{\hbar}\frac{e}{c}\Phi}$$

$\Rightarrow$ there is no change <u>only</u> if $\Phi_n = 2\pi\frac{\hbar c}{e}n$

flux quantization! $\Phi_n = n\,\phi_0$

$\phi_0 = \frac{hc}{e}$ <u>flux quantum</u>

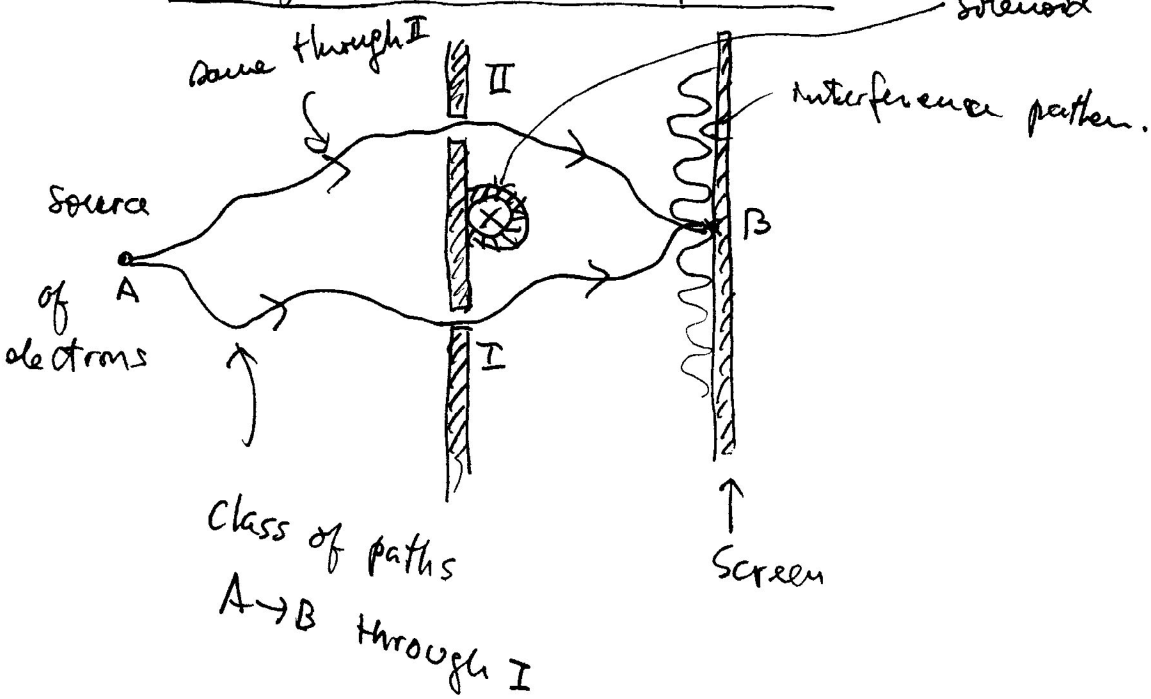
For <u>any</u> other Φ the amplitude depends on the path!

$\Rightarrow$ interference among paths going around $\vec{r}=0$

$\Rightarrow$ this is the <u>Bohm-Aharonov effect</u>.

Akip

Modified Double Slit Experiment



$$\langle B, t_f | A, t_i \rangle = \sum_{\substack{\text{paths} \\ \text{through I}}} e^{iS/\hbar} + \sum_{\substack{\text{paths} \\ \text{through II}}} e^{iS/\hbar}$$

But the paths have an action

$$S = \int_i^f dt \, \frac{1}{2} m \left(\frac{d\vec{v}}{dt}\right)^2 + \frac{e}{c} \int_i^f dt \, \vec{A} \cdot \vec{v} =$$

$$= \int_i^f dt \, \frac{1}{2} m \left(\frac{d\vec{r}}{dt}\right)^2 + \frac{e}{c} \int_A^B d\vec{r} \cdot \vec{A}$$

$$S_{II} - S_I = \int_{i\,II}^f dt \, \frac{1}{2} m \vec{v}^2 - \int_{i\,I}^f dt \, \frac{1}{2} m \vec{v}^2$$

$$+ \frac{e}{c} \int_{A\,II}^B d\vec{r} \cdot \vec{A} - \int_{A\,I}^B d\vec{r} \cdot \vec{A}$$

$$= \Delta S_0 + \frac{e}{c} \oint d\vec{r} \cdot \vec{A} = \Delta S_0 + \frac{e}{c} \Phi$$

$$\Rightarrow \langle B t_f | A t_i \rangle = \sum_I e^{iS_0/\hbar} + e^{i\frac{e}{\hbar c}\Phi} \sum_{II} e^{iS_0/\hbar}$$

The e.m. as a perturbation (classical e.m. field)

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + V(\vec{r}) + e\phi(\vec{r},t)$$

$$\vec{p} = \frac{\hbar}{i}\vec{\nabla}$$

$$\Rightarrow [p_i, A_j] = \frac{\hbar}{i}[\nabla_i, A_j] = \frac{\hbar}{i}\frac{\partial A_j}{\partial x_i} \neq 0$$

$$\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 = \left(\vec{p} - \frac{e}{c}\vec{A}\right)\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right) = \vec{p}^2 + \frac{e^2}{c^2}\vec{A}^2 - \frac{e}{c}\left(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}\right)$$

$$\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p} = \sum_i \{p_i, A_i\} = \sum_i [p_i, A_i] + 2\vec{A}\cdot\vec{p}$$

$$= \frac{\hbar}{i}\vec{\nabla}\cdot\vec{A} + 2\vec{A}\cdot\vec{p}$$

$\Rightarrow$ in the gauge $\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p} = 2\vec{A}\cdot\vec{p}$

For N particles

$$H = \sum_{i=1}^{N}\left[\frac{1}{2m_i}\left(\vec{p}_i - \frac{e_i}{c}\vec{A}(\vec{r}_i,t)\right)^2 + e_i\phi(\vec{r}_i,t) + V(\vec{r}_i)\right]$$

$$= H_0 + H_{int}$$

$$H_0 = \sum_{i=1}^{N}\left[\frac{\vec{p}_i^{\,2}}{2m_i} + V(\vec{r}_i)\right]$$

$$H_{int} = \sum_{i=1}^{N}\left[-\frac{e_i}{2m_i c}\left(\vec{p}_i\cdot\vec{A}(\vec{r}_i,t) + \vec{A}(\vec{r}_i,t)\cdot\vec{p}_i\right) + \frac{e_i^2}{2m_i c^2}\vec{A}^2(\vec{r}_i,t) + e_i\phi(\vec{r}_i,t)\right]$$

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<u>Def</u> : $\hat{\rho}(\vec{r}_i t) = \sum_{i=1}^{N} e_i \delta(\vec{r} - \vec{r}_i)$

$$\Rightarrow \sum_{i=1}^{N} e_i \, \phi(\vec{r}_i, t) = \int d^3r \; \phi(\vec{r}, t) \sum_{i=1}^{N} e_i \delta(\vec{r} - \vec{r}_i)$$

$$\equiv \int d^3r \; \rho(\vec{r}) \, \phi(\vec{r}, t)$$

$$\rho(\vec{r}) = \sum_{i=1}^{N} \delta(\vec{r} - \vec{r}_i) = \int \frac{d^3P}{(2\pi)^3} \sum_{i=1}^{N} e^{i P \cdot (\vec{r} - \vec{r}_i)}$$

where the $\vec{r}_i$ are operators.

$\rho(\vec{r})$ is the <u>density operator</u>. ($\phi(\vec{r})$ is <u>not</u> an op.)

$(m_i = m)$

<u>Def</u>: $$\vec{j}(\vec{r}) = \frac{1}{2} \sum_{i=1}^{N} \left(\frac{\vec{p}_i}{m_\xi} \delta(\vec{r} - \vec{r}_i) + \delta(\vec{r} - \vec{r}_i) \frac{\vec{p}_i}{m_\xi} \right)$$

$$\Rightarrow \sum_{i=1}^{N} -\frac{e}{2m_\xi c} \left(\vec{p}_i \cdot \vec{A}(\vec{r}_i, t) + \vec{A}(\vec{r}_i, t) \cdot \vec{p}_i \right) =$$

$$= -\int d^3r \; \frac{e}{c} \, \vec{j}(\vec{r}) \cdot \vec{A}(\vec{r}, t)$$

$\vec{j}(\vec{r})$ is Hermitean but not gauge invariant. It is the paramagnetic current.

$\vec{J}(\vec{r}) = \vec{j}(\vec{r}) - \frac{e}{m_\xi c} \vec{A}(\vec{r}, t) \, \rho(\vec{r})$ is both Hermitean and gauge invariant,

diamagnetic current

is the true <u>particle current</u> operator.

$$H_{int} = \int d^3r \left[-\frac{e}{c}\vec{j}(\vec{r})\cdot\vec{A}(\vec{r},t) + \frac{e^2}{2mc^2}\rho(\vec{r})\vec{A}^2(\vec{r},t) + \right.$$

$$\left. + e\rho(\vec{r})\phi(\vec{r},t) \right]$$

Absorption of Light

We will assume that $\vec{A}$ is typically "small" w.r.t. "atomic" fields (i.e. those that determine the energy levels of H_0). In this limit the diamagnetic effects are small ($\approx O(A^2)$)

$$\Rightarrow H_{int} \simeq -\frac{e}{c}\int d^3r \; \vec{j}(\vec{r})\cdot\vec{A}(\vec{r},t)$$

$$\vec{A}(\vec{r},t) = \sum_{\vec{k},\vec{\lambda}} \left[\vec{\lambda}\,\frac{A(\vec{k},\vec{\lambda})}{\sqrt{V}}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \frac{\vec{\lambda}^*\, A(\vec{k},\vec{\lambda})^*}{\sqrt{V}}\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \right]$$

$$\vec{j}(\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\, \vec{j}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}} \quad , \quad \vec{\lambda} = \vec{\lambda}(\vec{k}) \,/\, \vec{k}\cdot\vec{\lambda} = 0$$

$$\Rightarrow H_{int} = -\frac{e}{c}\sum_{\vec{k},\vec{\lambda}} \left[A(\vec{k},\vec{\lambda})\, \vec{j}(-\vec{k})\cdot\vec{\lambda}\, \frac{e^{-i\omega t}}{\sqrt{V}} + \right.$$

$$\left. + A(\vec{k},\vec{\lambda})^*\, \vec{j}(\vec{k})\cdot\vec{\lambda}^*\, \frac{e^{i\omega t}}{\sqrt{V}} \right]$$

$$\vec{j}(\vec{k}) = \int d^3r\, e^{-i\vec{k}\cdot\vec{r}}\, \vec{j}(\vec{r}) \equiv \frac{1}{2}\sum_{i=1}^{N} \left(\frac{\vec{p}_i}{m}\, e^{-i\vec{k}\cdot\vec{r}_i} + e^{-i\vec{k}\cdot\vec{r}_i}\, \frac{\vec{p}_i}{m} \right)$$

We will use the Golden Rule to compute the transition probab. even for the discrete spectrum (which is correct if there is a continuum of frequencies in the incident light).

Absorption: upward transitions caused by the positive frequencies in H_{int}. If the atom is in state $|0\rangle$ with energy $\mathcal{E}_0$ at $t_i \to -\infty \Rightarrow$ transition rate is

$$\Gamma^{abs}_{0\to n;\vec{k}\vec{\lambda}} = \frac{2\pi}{\hbar}\,\delta(\mathcal{E}_n - \mathcal{E}_0 - \hbar\omega)\,\frac{e^2}{V c^2}\,|A(\vec{k},\vec{\lambda})|^2 \cdot |\langle n|\vec{j}(-k)\cdot\vec{\lambda}|0\rangle|^2$$

atom

$\hbar\omega = \hbar c|\vec{k}|$

total rate $0 \to n$

$$\Gamma^{abs}_{0\to n} = \sum_{\vec{k},\lambda} \Gamma^{abs}_{0\to n;\vec{k}\vec{\lambda}} =$$

$$= \frac{1}{V}\sum_{\vec{k},\vec{\lambda}} \frac{2\pi}{\hbar}\,\delta(\mathcal{E}_n - \mathcal{E}_0 - \hbar\omega)\,\frac{e^2}{c^2}|A(\vec{k},\vec{\lambda})|^2 \cdot |\langle n|\vec{j}(-\vec{k})\cdot\vec{\lambda}|0\rangle|^2$$

$$\frac{1}{V}\sum_{\vec{k}} \equiv \int \frac{d^3k}{(2\pi)^3} \equiv \int \frac{\omega^2 d\omega\, d\Omega}{(2\pi c)^3}$$

$$\Gamma^{abs}_{0\to n} = \frac{2\pi e^2}{\hbar^2 c^2} \frac{\omega^2}{(2\pi c)^3} \int d\Omega \sum_{\vec\lambda} \left| \langle n | \vec{j}(-\vec{k}) \cdot \vec{\lambda} | 0 \rangle \right|^2$$

$$\omega = \frac{\mathcal{E}_n - \mathcal{E}_0}{\hbar}$$

<u>Incident Beam:</u> polarized with pol. vectr. $\vec\lambda$ and subtends a solid angle $d\Omega$

$\Rightarrow$ rate of energy transport in the beam (Poynting vector)

is $\displaystyle \frac{1}{V} \sum_{\vec k} \frac{\omega^2}{2\pi c} |A(\vec k, \vec\lambda)|^2 = d\Omega \int d\omega \frac{\omega^4}{(2\pi c)^4} |A(\vec k, \vec\lambda)|^2$

$\Rightarrow$ the intensity $\displaystyle I(\omega) = d\Omega \frac{\omega^4}{(2\pi c)^4} |A(\vec k, \vec\lambda)|^2$

$[I] =$ ergs/cm^2-rad

$$\Rightarrow \boxed{\Gamma^{abs}_{0\to n} = \frac{4\pi^2 e^2}{\hbar^2 c\, \omega^2} I(\omega) \left| \langle n | \vec{j}(-\vec{k}) \cdot \vec{\lambda} | 0 \rangle \right|^2}$$

(b) Rate of <u>downward transition</u> $\Rightarrow$ induced emmission.

$$\Gamma^{ind.\,emm}_{n\to 0} = \frac{1}{V} \sum_{\vec k, \vec\lambda} \frac{2\pi}{\hbar} \delta(\mathcal{E}_n - \mathcal{E}_0 - \hbar\omega) \frac{e^2}{c^2} |A(\vec k, \vec\lambda)|^2 \times$$
$$\times \left| \langle 0 | \vec{j}(\vec{k}) \cdot \vec{\lambda}^* | n \rangle \right|^2$$

stimulated emmission.

$$\omega = \frac{\mathcal{E}_n - \mathcal{E}_0}{\hbar}$$

$$\boxed{\Gamma^{abs}_{0\to n} = \Gamma^{ind.\,emm.}_{n\to 0}}$$

$\Rightarrow$ no <u>spontaneous emmission</u>!

Since $\langle 0 | \vec{j}(-\vec{k}) \cdot \vec{\lambda} | n \rangle = \langle n | \vec{j}(\vec{k}) \cdot \vec{\lambda}^* | 0 \rangle^*$

In terms of photons, each has energy $\hbar\omega$, ~~and~~ momentum $\vec{k}$ and polarization $\vec{\lambda}$ $\Rightarrow$ Energy of the beam

$$E = \sum_{\vec{k},\lambda} \hbar\omega \, N(\vec{k},\vec{\lambda})$$

↳ # of photons in mode $(\vec{k},\vec{\lambda})$

$$\Rightarrow \hbar\omega \, N(\vec{k},\vec{\lambda}) = \frac{\omega^2}{2\pi c^2} |A(\vec{k},\vec{\lambda})|^2$$

$$|A(\vec{k},\vec{\lambda})|^2 = \frac{2\pi\hbar c^2}{\omega} N(\vec{k},\vec{\lambda})$$

$$\Rightarrow \Gamma^{abs}_{0\to n} = \Gamma^{s.e.}_{n\to 0} = \sum_{\vec{k},\vec{\lambda}} \frac{4\pi^2 e^2}{\omega V} \delta(\mathcal{E}_n - \mathcal{E}_0 - \hbar\omega) \left|\langle n| \vec{j}(-\vec{k})\cdot\vec{\lambda} |0\rangle\right|^2$$

$N(\vec{k},\vec{\lambda})$

[comment on incoherence].

Total absorption rate of photons in mode $(\vec{k},\vec{\lambda})$:

$$\Gamma^{abs}(\omega) = \frac{c\,N(\vec{k},\vec{\lambda})}{V} \frac{4\pi^2 e^2}{\omega c} \sum_n |\langle n| \vec{j}(-\vec{k})\cdot\vec{\lambda} |0\rangle|^2 \delta(\mathcal{E}_n - \mathcal{E}_0 - \hbar\omega)$$

$\frac{N(\vec{k},\vec{\lambda})}{V} \equiv$ photon density $\Rightarrow \frac{c\,N(\vec{k},\vec{\lambda})}{V}$ incident photon flux

Absorption cross section $= \dfrac{\Gamma^{abs}(\omega)}{\text{photon flux}} = \sigma_{abs}(\omega)$

$$\sigma_{abs}(\omega) = \frac{4\pi^2 e^2}{\omega c} \sum_n |\langle n| \vec{j}(-k)\cdot\vec{\lambda} |0\rangle|^2 \, \delta(\mathcal{E}_n - \mathcal{E}_0 - \hbar\omega)$$

Multipole Radiation

We will now discuss briefly the application of these ideas to the multipole radiation in spontaneous emission of atoms. I will just sketch the calculations. ~~also~~ Details are in Baym p. 376 – 380.

The energy eigenstates of an atom are not eigenstates of orbital angular momentum due to the spin-orbit interaction, but are eigenstates of the total angular momentum (orbital + spin)

The amplitude of emission of a photon of wave vector $\vec{k}$ and polarization $\vec{\varepsilon}(\vec{k},\nu)$, while the atom makes the transition from $|i\rangle = |\lambda j m\rangle \to |f\rangle = |\bar{\lambda}\bar{j}\bar{m}\rangle$ is determined by

$$\langle \lambda j m | \vec{J}(-\vec{k}) \cdot \vec{\varepsilon}(\vec{k},\nu) | \bar{\lambda}\bar{j}\bar{m}\rangle$$

$$\vec{J}(-\vec{k}) = \int d^3r \; e^{i\vec{k}\cdot\vec{r}} \; \underbrace{\vec{J}(\vec{r})}_{\text{current operator}}$$

We will now express this amplitude in such a way that, through the Wigner-Eckart thm., we can read off the selection rules.

$$e^{i\vec{k}.\vec{r}} = 4\pi \sum_{l,m} i^l \, Y_{lm}(\Omega_k)^* \, Y_{lm}(\Omega_r) \, j_l(kr)$$

is the expansion in spherical harmonics.

with $j_l(kr) = \sqrt{\frac{\pi}{2kr}} \, J_{l+\frac{1}{2}}(kr)$ ← Bessel functions.

$$\Rightarrow \vec{\varepsilon} \cdot \vec{J}(-\vec{k}) = 4\pi \sum_{lm} i^l \, Y_{lm}(\Omega_k)^* \int d^3r \, Y_{lm}(\Omega_r) \, j_l(kr) \, \vec{J}(r) \cdot \vec{\varepsilon}$$

In spherical (α q) components

$$\vec{\varepsilon} \cdot \vec{J}(\vec{r}) = \sum_{q} (-1)^q \, \varepsilon_{-q} \, J_q(\vec{r}) = \sum_q \varepsilon_q^* \, J_q(\vec{r})$$

$$\vec{\varepsilon} \cdot \vec{J}(-k) = 4\pi \sum_{qlm} i^l \left(Y_{lm}(\Omega_k)\, \varepsilon_q\right)^* \int d^3r \, Y_{lm}(\Omega_r) \, j_l(kr) \, J_q(r)$$

CGC's orthogonality $\Rightarrow$ $\equiv 4\pi \sum_{ll'm'} i^l \, \Phi^{(l')}_{m'}(l)^* \, \mathcal{J}^{(l')}_{m'}(l)$

$$\mathcal{J}^{(l')}_{m'}(l) = \sum_{q''m''} \int d^3r \, Y_{lm''}(\Omega_r) \, j_l(kr) \, J_{q''}(\vec{r}) \, \langle 1 l q'' m'' | 1 l l' m' \rangle$$

$$\Phi^{(l')}_{m'}(l) = \Phi^{(l')}_{m'}(l, \Omega_k, \vec{\varepsilon}) = \sum_{qm} Y_{lm}(\Omega_k) \, \varepsilon_q^* \langle 1 l q m | 1 l l' m' \rangle$$

$\mathcal{J}^{(l')}_{m'}(l)$ is an irreducible tensor of order l' (m'th component)

From the CGC' $\langle 1 l q'' m'' | 1 l l' m' \rangle \Rightarrow l = l', l' \pm 1$
and $q'' + m'' = m'$

For $ka_0 \ll 1 \Rightarrow j_l(kr) \approx (kr)^l / (2l+1)!!$

For $l = l'$ $\Rightarrow$ $\mathcal{J}^{(l')}_{m'}(l') \sim k^{l'} \times$ magnetic $2^{l'}$-pole operator.

For $l = l'-1$ $\Rightarrow$ $\mathcal{J}^{(l')}_{m'}(l'-1) \sim k^{l'-1} \times$ electric $2^{l'}$-pole

$l = l'+1$ $\quad \mathcal{J}^{(l')}_{m'}(l'+1) \sim k^{l'+1} \times$ electric $2^{l'}$-pole (usually very small)

e.g. $$\mathcal{J}^{(1)}_{m'}(1) = \sum_{qm} \int d^3r \, \frac{kr}{3} \, Y_{1m}(\Omega_r) \, J_q(r) \langle 11qm | 111m' \rangle$$

$(l = l' = 1)$ $$r \, Y_{1m}(\hat{e}_r) = r_m \sqrt{\frac{3}{4\pi}}$$

$$\Rightarrow \sum_{qm} r_m J_q(\vec{r}) \langle 11qm | 111m' \rangle = \frac{(\vec{r} \times \vec{J}(\vec{r}))_{m'}}{i\sqrt{2}}$$

$$\Rightarrow \mathcal{J}^{(1)}_{m'}(1) = \frac{k}{i\sqrt{24\pi}} \int d^3r \, (\vec{r} \times \vec{J}(\vec{r}))_{m'}$$

magnetic dipole.

$\Rightarrow \sum_{l'}$ is a sum over diff. $2^{l'}$-pole transitions.

$$\langle \lambda j m | \vec{J}(-\vec{k}) \cdot \vec{\varepsilon} | \bar{\lambda} \bar{j} \bar{m} \rangle = \sum_{ll'm'} 4\pi i^l \, \Phi^{(l')}_{m'}(l)^* \langle \lambda j m | \mathcal{J}^{(l')}_{m'}(l) | \bar{\lambda} \bar{j} \bar{m} \rangle$$

$$\underset{\text{Wigner Eckart}}{=} \sum_{ll'm'} 4\pi i^l \, \Phi^{(l')}_{m'}(l)^* \, \frac{\langle \lambda j \| \mathcal{J}^{(l')}_{m'}(l) \| \bar{\lambda} \bar{j} \rangle}{\sqrt{2j+1}} \langle l' \bar{j} m' \bar{m} | l' \bar{j} j m \rangle$$

↑ here is the dynamics

$\Rightarrow$ the CGC $\Rightarrow$ $m = \bar{m} + m'$

and $|l' - j| \le \bar{j} \le l' + j$

or $|l' - \bar{j}| \le j \le l' + \bar{j}$ are the allowed transitions.

Parity imposes additional restrictions since $P\,\vec{J}(r)P^{-1} = -\vec{J}(-\vec{r})$

$\Rightarrow$ if the states $|\lambda j m\rangle$, $|\bar{\lambda}\bar{j}\bar{m}\rangle$ have def. parity

$\Rightarrow$ $\langle \lambda j \| \mathcal{J}^{(\ell')}(\ell) \| \bar{\lambda}\bar{j}\rangle = 0$ unless the final parity is $(-1)^{\ell+1}$ the initial parity.

For an electric dipole, $\ell = 0$, $\ell' = 1$ $\Rightarrow$ $\bar{j} = j \pm 1$ or $\bar{j} = j$

$\Rightarrow$ parity of the final state = − parity of initial state.

This reduces ~~#~~ to our previous analysis.